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On nonperturbative calculations in quantum electrodynamics

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Abstract. A new approach to nonperturbative calculations in quantum electrodynamics is proposed. This approach is based on a regular iteration scheme for the solution of Schwinger–Dyson equations for generating the functional of Green functions. The approach allows one to take into account the gauge invariance conditions (Ward identities) and to perform the renormalization programme.

The iteration scheme can be realized in two versions. The first one ('perturbative vacuum') corresponds to chain summation in the diagram language. In this version the exact result of a two-dimensional Schwinger model is reproduced as the first step in the calculations, but in four-dimensional theory the nonphysical singularity (Landau pole) arises which leads to the triviality of the renormalized theory. The second version ('nonperturbative vacuum') corresponds to ladder summation and permits one to perform nonperturbative calculations of physical quantities in spite of the triviality problem.

For a chiral-symmetrical leading approximation two terms of the expansion of the first-step vertex function over photon momentum are calculated. The formula $f_2 = \alpha/(2\pi - \alpha)$ for the anomalous magnetic moment is obtained (α is the fine structure constant).

For a linearized equation of the leading approximation a problem of dynamical chiral symmetry breaking is considered and the calculations are performed for renormalized theory in Minkowski space. In the strong coupling region $\alpha \geqslant \pi/3$ the results correspond to earlier investigations performed in Euclidean theory with cutoff: solutions arise with the breakdown of chiral symmetry. For the renormalized theory a solution with breakdown of chiral symmetry is also possible in the weak coupling region $\alpha < \pi/3$, but with a subsidiary condition on the value of α which follows from the gauge invariance.

1. Introduction

The problem of nonperturbative calculations in quantum electrodynamics (QED) arose practically simultaneously with the principal solution of the problem of perturbative calculations which was based on renormalized coupling constant perturbation theory. It is necessary to recognize, however, that the progress in nonperturbative calculations during the last few decades has not been too great. A quantitative description of nonperturbative effects is either based on nonrelativistic foundations (an example is the bound state description based on the nonrelativistic Coulomb problem) or is rather open to criticism. In addition, the problem of the inner inconsistency of QED exists (see [1]). This problem can be formulated as a deep-rooted thesis on the triviality of QED in the nonperturbative region (see, e.g., [2, 3] and references therein).

This triviality means that the only noncontradictory value of the renormalized coupling is zero. The absence of asymptotic freedom in QED and unsuccessful attempts to find another

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type of self-consistent ultraviolet behaviour are strong arguments in favour of the triviality†. An extremal expression of this point of view is the statement that QED can be treated exceptionally as the renormalized coupling constant perturbation theory and, consequently, any nonperturbative calculations are excluded from the consideration. Such a situation is not desperate since at very high energies the QED becomes a part of a grand unification theory which is based on non-Abelian gauge theory with self-consistent asymptotically free ultraviolet behaviour. However, it sounds rather unnatural that, for a consistent relativistic calculation of positronium energy levels, for example, one should exploit a grand unification theory, while it is not needed for calculations of annihilation cross sections. It is difficult to find any physical reason for such a major difference in these problems. Therefore the possibility of performing approximate nonperturbative calculations of physical quantities in the framework of QED itself without contradictions with the triviality of the exact theory (as in perturbation theory) seems to be a necessary component of the theory.

At first sight the main problem of nonperturbative calculations in QED is the absence of a universal small parameter in addition to the fine structure constant. For this reason, any partial summation of a perturbative series seems to be an arbitrary procedure which can be accepted only by the physical meaning of the results. At the same time the absence of a small parameter itself is not an obstacle for using some approximation. A motivated approximation can give quite satisfactory results, even without an explicit small parameter: examples are applications of variational methods or mean field type approximations in different areas of physical theory.

A general problem facing various nonperturbative approximations in QED is consistently taking into account requirements of gauge invariance and renormalizability. It is clear that a necessary condition for such a requirement is the existence of a regular iteration scheme which, in principle, permits one to make an arbitrarily large number of steps toward an exact solution to the problem. In this paper such a scheme is proposed. The scheme is based on an approximation of the Schwinger–Dyson equations (SDEs) for the generating functional of Green functions in QED by an exactly soluble equation. Its solution generates a linear iteration scheme, each step of which is described by a closed system of integro-differential equations. The requirements of gauge invariance (in the form of Ward identities) are easily taken into account at each step of the iteration. The renormalization of equations at each step is also not a major problem. Note that equations for Green functions at the leading approximation and at the first step of the iterations appear to be familiar. Similar equations were written and investigated earlier in other contexts. A new concept is its appearance in the structure of the regular iteration scheme, and it is this circumstance that allows one to give them a successive quantum-field-theoretical interpretation.

Using a bilocal fermion source gives a possibility of formulating the iteration scheme in two versions. First, in the language of Feynman diagrams of perturbation theory is an analogue of the summation of chain diagrams with a fermion loop. This version is called 'calculations over perturbative vacuum' since a unique connected Green function of the leading (vacuum) approximation is the free-electron propagator. The calculations over perturbative vacuum for two-dimensional electrodynamics give the exact result of the Schwinger model for a photon propagator as early as the first step, but for a physical four-dimensional case they lead to the appearance a nonphysical Landau pole in the photon propagator and, as a consequence, to the triviality of the renormalized theory. Thus the practical meaning of the calculations over perturbative vacuum for four-dimensional QED becomes equal to zero, though they produce the main nonperturbative effect—the triviality of the full theory.

The second version of the iteration scheme can be compared with the diagram language

† In QED with dynamical chiral symmetry breaking (DCSB) the situation can be different (see [4, 5]).

with a ladder summation. This version is called 'calculations over nonperturbative vacuum' since the electron propagator of the leading vacuum approximation is a solution of a nontrivial nonlinear equation. For this version of the iteration scheme the nonperturbative calculations become possible without contradiction to the triviality of the full theory. The main content of this paper is devoted to the investigation of this version. Apart from the formulation of general states and the principles of renormalization of Green functions, this paper contains the calculation of the first-step vertex function (in the case of a chiral-symmetric solution of the leading approximation equation). In addition, for a linearized version of the theory the problem of DCSB is investigated.

The structure of the paper is as follows: in the first section the necessary notation and definitions are given; SDEs for the generating functional of Green functions are introduced in the formalism of a bilocal fermion source, and generating relations for Ward identities are considered. In the second section a general construction of the iteration scheme for the solution of SDEs is given; in addition, the leading approximation and the first step of iteration over the perturbative vacuum are considered. In section 3 the renormalization of the first-step equations is performed; it is demonstrated that, for the four-dimensional theory, the calculations over perturbative vacuum lead to the trivial theory already at the first step. In section 4 a scheme for calculations over nonperturbative vacuum is formulated. In section 5 two first terms of expansion of the vertex function in photon momentum are calculated for a chiral-symmetric vacuum. These calculations allow one to obtain a simple formula for an anomalous magnetic moment: $f_2 = \alpha/(2\pi - \alpha)$, where α is the fine structure constant. Section 6 is devoted to investigation of DCSB in QED. In the conclusions a discussion of these results is given.

2. SDEs and Ward identities

We shall consider the theory of a spinor field $\psi(x)$ (electron) interacting with an Abelian gauge field $A_\mu(x)$ (photon) in n-dimensional Minkowski space with the metrics $x^2 \equiv x_\mu x_\mu = x_0^2 - x_1^2 - \cdots - x_{n-1}^2$. (For notational simplicity we write all vector indices as subscripts.) The Lagrangian with a gauge fixing term has the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{2d_l}(\partial_\mu A_\mu)^2 + \bar{\psi}(\mathrm{i}\hat{\partial} - m + e\hat{A})\psi. \tag{1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\hat{A} \equiv A_\mu\gamma_\mu$, $\bar{\psi} = \psi^*\gamma_0$, m is the mass of an electron, e is a charge (coupling constant), d_l is a gauge parameter and γ_μ are Dirac matrices.

A generating functional of Green functions (vacuum expectation values for the T product of fields) can be represented as a functional integral:

$$G(J,\eta) = \int D(\psi,\bar{\psi},A)\exp\mathrm{i}\Big\{\int \mathrm{d}x\,(\mathcal{L} + J_\mu(x)A_\mu(x)) - \int \mathrm{d}x\,\mathrm{d}y\,\bar{\psi}^\beta(y)\eta^{\beta\alpha}(y,x)\psi^\alpha(x)\Big\}. \tag{2}$$

Here $J_\mu(x)$ is a source of the gauge field and $\eta^{\beta\alpha}(y,x)$ is a bilocal source of the spinor field (α and β are spinor indices). The normalization constant is omitted.

Functional derivatives of G with respect to sources are vacuum expectation values:

$$\frac{\delta G}{\delta J_\mu(x)} = \mathrm{i}\langle 0|A_\mu(x)|0\rangle \qquad \frac{\delta G}{\delta \eta^{\beta\alpha}(y,x)} = \mathrm{i}\langle 0|T\{\psi^\alpha(x)\bar{\psi}^\beta(y)\}|0\rangle. \tag{3}$$

A heuristic derivation of SDEs for the generating functional G is based on the following relations (see, e.g., [6, 7]):

$$0 = \int D(\psi, \bar{\psi}, A) \frac{\delta}{\delta A_\mu(x)} \exp \mathrm{i} \left\{ \int \mathrm{d}x \, (\mathcal{L} + J_\mu(x) A_\mu(x)) - \int \mathrm{d}x \, \mathrm{d}y \, \bar{\psi}(y) \eta(y, x) \psi(x) \right\} \tag{4}$$

$$0 = \int D(\psi, \bar{\psi}, A) \frac{\delta}{\delta \bar{\psi}(x)} \bar{\psi}(y) \exp \mathrm{i} \left\{ \int \mathrm{d}x \, (\mathcal{L} + J_\mu(x) A_\mu(x)) - \int \mathrm{d}x \, \mathrm{d}y \, \bar{\psi}(y) \eta(y, x) \psi(x) \right\}. \tag{5}$$

Taking derivatives of equations (4) and (5), and taking into account equation (3), we get SDEs for the generating functional of the Green functions of QED:

$$\left(g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu + \frac{1}{d_l} \partial_\mu \partial_\nu \right) \frac{1}{\mathrm{i}} \frac{\delta G}{\delta J_\nu(x)} + \mathrm{i} e \, \mathrm{tr} \left\{ \gamma_\mu \frac{\delta G}{\delta \eta(x, x)} \right\} + J_\mu(x) G = 0 \tag{6}$$

$$\delta(x - y) G + (\mathrm{i} \hat{\partial} - m) \frac{\delta G}{\delta \eta(y, x)} + \frac{e}{\mathrm{i}} \gamma_\mu \frac{\delta^2 G}{\delta J_\mu(x) \delta \eta(y, x)} - \int \mathrm{d}x' \, \eta(x, x') \frac{\delta G}{\delta \eta(y, x')} = 0. \tag{7}$$

(Here and in what follows ∂_μ denotes differentiation with respect to variable x, while differentiation with respect to another variable will be denoted by indication of this variable as a superscript. For example, the differentiation in variable y will be denoted as ∂_μ^y.)

We consider in this and the following sections the unrenormalized theory. Therefore for all singular divergent quantities some regularization is implied. A renormalization of SDEs is considered below (section 3).

Gauge invariance imposes some limitations on the solutions of SDEs (6) and (7), known as Ward identities. Acting by ∂_μ on equation (6), we get a relation

$$\frac{1}{d_l} \partial^2 \partial_\mu \frac{1}{\mathrm{i}} \frac{\delta G}{\delta J_\mu(x)} + e \, \mathrm{tr} \left\{ \mathrm{i} \hat{\partial} \frac{\delta G}{\delta \eta(x, x)} \right\} + \partial_\mu J_\mu(x) G = 0 \tag{8}$$

which we shall call the first generating relation. One more relation follows from equation (7) and its conjugated SDE:

$$\delta(x - y) G + \frac{\delta G}{\delta \eta(y, x)} (-\mathrm{i} \hat{\partial}^y - m) + \frac{e}{\mathrm{i}} \frac{\delta^2 G}{\delta J_\mu(y) \delta \eta(y, x)} \gamma_\mu - \int \mathrm{d}x' \frac{\delta G}{\delta \eta(x', x)} \eta(x', y) = 0. \tag{9}$$

Equation (9) is a consequence of the relation

$$0 = \int D(\psi, \bar{\psi}, A) \frac{\delta}{\delta \psi(y)} \psi(x) \exp \mathrm{i} \left\{ \int \mathrm{d}x \, (\mathcal{L} + J_\mu(x) A_\mu(x)) - \int \mathrm{d}x \, \mathrm{d}y \, \bar{\psi}(y) \eta(y, x) \psi(x) \right\} \tag{10}$$

which is conjugated to (5). We subtract equation (9) from (7), take a trace over spinor indices and put $y = x$. As a result of these simple manipulations we get the relation

$$\mathrm{tr} \left\{ \mathrm{i} \hat{\partial} \frac{\delta G}{\delta \eta(x, x)} \right\} = \int \mathrm{d}x' \, \mathrm{tr} \left\{ \eta(x, x') \frac{\delta G}{\delta \eta(x, x')} - \frac{\delta G}{\delta \eta(x', x)} \eta(x', x) \right\} \tag{11}$$

which we shall call the second generating relation. Relation (11) is fulfilled for numerous classes of interactions local with respect to fermions (see [8]). For QED we can combine both relations into one, substituting $\mathrm{tr}\{\mathrm{i} \hat{\partial} \frac{\delta G}{\delta \eta(x, x)}\}$ from (11) into (8). As a result we get

$$\frac{\mathrm{i}}{d_l} \partial^2 \partial_\mu \frac{\delta G}{\delta J_\mu(x)} = \partial_\mu J_\mu(x) G + e \int \mathrm{d}x' \, \mathrm{tr} \left\{ \eta(x, x') \frac{\delta G}{\delta \eta(x, x')} - \frac{\delta G}{\delta \eta(x', x)} \eta(x', x) \right\} \tag{12}$$

which is the generating relation of Ward identities for QED.

Differentiating equation (12) with respect to J_λ and switching off the sources we obtain the familiar relation

$$\partial^2 \partial_\mu D_{\mu\lambda}(x-y) = d_l \partial_\lambda \delta(x-y) \tag{13}$$

where

$$D_{\mu\lambda}(x-y) \equiv \mathrm{i}\left.\frac{\delta^2 G}{\delta J_\lambda(y)\delta J_\mu(x)}\right|_{J=\eta=0} \tag{14}$$

is the photon propagator. From relation (13) it follows that the longitudinal part of the full photon propagator (in momentum space) is as follows:

$$D^{\mathrm{long}}_{\mu\lambda}(k) = -d_l \frac{k_\mu k_\lambda}{(k^2)^2}. \tag{15}$$

Differentiating equation (12) with respect to η, we obtain, after switching off the sources, another familiar relation:

$$\frac{\mathrm{i}}{d_l}\partial^2 \partial_\mu F_\mu(x; x', y') = e[\delta(x-y') - \delta(x-x')]S(x'-y'). \tag{16}$$

Here

$$F_\mu(x; x', y') \equiv \left.\frac{\delta^2 G}{\delta J_\mu(x)\delta\eta(y', x')}\right|_{J=\eta=0} \tag{17}$$

is the three-point function and

$$S(x-y) \equiv \left.\frac{\delta G}{\delta\eta(y,x)}\right|_{J=\eta=0} \tag{18}$$

is the full electron propagator.

Relation (16) looks much more familiar if one goes to the truncated three-pointer (vertex function) Γ_μ which is defined as

$$\Gamma_\mu(x \mid x', y') \equiv \int \mathrm{d}x_1\, \mathrm{d}x_1'\, \mathrm{d}y_1'\, S^{-1}(x'-x_1')F_\nu(x_1; x_1', y_1')S^{-1}(y_1'-y')D^{-1}_{\nu\mu}(x_1-x). \tag{19}$$

Exploiting this definition and equation (13), we obtain the Ward identity for a vertex function:

$$\mathrm{i}\partial_\mu \Gamma_\mu(x \mid x', y') = e[\delta(x-x') - \delta(x-y')]S^{-1}(x'-y'). \tag{20}$$

3. Iteration scheme

At $e = 0$ SDEs (6) and (7) have a solution

$$G^{\mathrm{free}} = \exp\left\{\frac{1}{2\mathrm{i}} J_\mu \star D^c_{\mu\nu} \star J_\nu + \mathrm{Tr}\log(1 + S^c \star \eta)\right\}$$

where

$$D^c_{\mu\nu} = \left[g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{d_l}\partial_\mu\partial_\nu\right]^{-1} \qquad S^c = (m - \mathrm{i}\hat{\partial})^{-1}$$

are free-field propagators, and the sign $\star$ denotes a multiplication in the operator sense. (Also in the operator sense we shall use Tr, as distinct from tr, which corresponds to the trace over spinor indices.) The functional G^{free} is a generating functional of Green functions for free fields and is a base for an iteration scheme of perturbation theory in the coupling constant e.

For solution of SDEs (6) and (7) we shall use another iteration scheme proposed in [9,10]. A general idea of this scheme is an approximation of functional-differential SDEs (6) and (7)

by equations with 'constant', i.e. independent of the sources J_μ and η, coefficients. Thus we approximate functional-differential SDEs near the point $J_\mu = 0, \eta = 0$. Since the objective of these calculations is to obtain Green functions, i.e. derivatives of G in zero, such an approximation seems to be quite natural. A circumstance of no small importance is a simplicity of the leading approximation of this scheme. Equations for all subsequent approximations can also be easily written. Green functions of every order of this scheme are defined as solutions of a closed system of equations. Technically this scheme is not much more complicated in comparison with the coupling constant perturbation theory, but in contrast to the latter, it maintains information which is inaccessible in any finite order of perturbation theory, i.e. it is a nonperturbative method.

In correspondence with the aforementioned, we choose as the leading approximation equations a system of functional-differential equations

$$\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{d_l}\partial_\mu\partial_\nu\right)\frac{1}{\mathrm{i}}\frac{\delta G^{(0)}}{\delta J_\nu(x)} + \mathrm{i}e\,\mathrm{tr}\left\{\gamma_\mu\frac{\delta G^{(0)}}{\delta\eta(x,x)}\right\} = 0 \tag{21}$$

$$\delta(x-y)G^{(0)} + (\mathrm{i}\hat{\partial} - m)\frac{\delta G^{(0)}}{\delta\eta(y,x)} + \frac{e}{\mathrm{i}}\gamma_\mu\frac{\delta^2 G^{(0)}}{\delta J_\mu(x)\delta\eta(y,x)} = 0. \tag{22}$$

A solution to the leading approximation equations (21), (22) is the functional

$$G^{(0)} = \exp\{\mathrm{i}V_\mu \star J_\mu + \mathrm{Tr}\,S^{(0)} \star \eta\} \tag{23}$$

where $V_\mu(x)$ and $S^{(0)}(x, y)$ satisfy equations

$$\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{d_l}\partial_\mu\partial_\nu\right)V_\nu(x) + \mathrm{i}e\,\mathrm{tr}\{\gamma_\mu S^{(0)}(x,x)\} = 0 \tag{24}$$

$$\delta(x-y) + (\mathrm{i}\hat{\partial} - m)S^{(0)}(x,y) + e\hat{V}(x)S^{(0)}(x,y) = 0. \tag{25}$$

By analogy with the theory of ordinary differential equations we shall call these equations characteristic ones.

In correspondence with the choice of leading approximation, the ith term of the iteration expansion of the generating functional

$$G = G^{(0)} + G^{(1)} + \cdots + G^{(i)} + \cdots \tag{26}$$

is a solution of the iteration scheme equations

$$\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{d_l}\partial_\mu\partial_\nu\right)\frac{1}{\mathrm{i}}\frac{\delta G^{(i)}}{\delta J_\nu(x)} + \mathrm{i}e\,\mathrm{tr}\left\{\gamma_\mu\frac{\delta G^{(i)}}{\delta\eta(x,x)}\right\} = -J_\mu(x)G^{(i-1)} \tag{27}$$

$$\delta(x-y)G^{(i)} + (\mathrm{i}\hat{\partial} - m)\frac{\delta G^{(i)}}{\delta\eta(y,x)} + \frac{e}{\mathrm{i}}\gamma_\mu\frac{\delta^2 G^{(i)}}{\delta J_\mu(x)\delta\eta(y,x)} = \int \mathrm{d}x'\,\eta(x,x')\frac{\delta G^{(i-1)}}{\delta\eta(y,x')}. \tag{28}$$

A solution of equations (27) and (28) is searched for in the form

$$G^{(i)} = P^{(i)}G^{(0)}. \tag{29}$$

On taking into account the characteristic equations (24) and (25) a system of equations for $P^{(i)}$ assumes the form

$$\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{d_l}\partial_\mu\partial_\nu\right)\frac{1}{\mathrm{i}}\frac{\delta P^{(i)}}{\delta J_\nu(x)} + \mathrm{i}e\,\mathrm{tr}\left\{\gamma_\mu\frac{\delta P^{(i)}}{\delta\eta(x,x)}\right\} = -J_\mu(x)P^{(i-1)} \tag{30}$$

$$(\mathrm{i}\hat{\partial} - m)\frac{\delta P^{(i)}}{\delta\eta(y,x)} + \frac{e}{\mathrm{i}}\gamma_\mu\frac{\delta^2 P^{(i)}}{\delta J_\mu(x)\delta\eta(y,x)} + e\hat{V}(x)\frac{\delta P^{(i)}}{\delta\eta(y,x)} + \frac{e}{\mathrm{i}}\gamma_\mu S^{(0)}(x,y)\frac{\delta P^{(i)}}{\delta J_\mu(x)}$$
$$= \int \mathrm{d}x'\,\eta(x,x')\left\{\frac{\delta P^{(i-1)}}{\delta\eta(y,x')} + S^{(0)}(x',y)P^{(i-1)}\right\}. \tag{31}$$

Since $P^{(0)} \equiv 1$, it is evident that, for any i, the functional $P^{(i)}$ is a polynomial in functional variables J and η. This situation is very important since it means the system of equations for coefficient functions of this functional must be closed in any order of the iteration scheme.

This iteration scheme has no explicit small parameter. In some sense, the sources J and η play the role of such a parameter. The expansion (26) of the generating functional should be treated as an approximation of $G(J, \eta)$ near the point $J_\mu = 0$, $\eta = 0$. In essence, instead of the question about the small parameter, one should pose a question regarding the convergence of the iteration series. Studying the convergence properties of the expansion for the quantum-field-theoretical case is obviously a very complicated problem, and we have no rigorous statement concerning this question. Nevertheless, some qualitative reasons can be given in favour of the following supposition: the convergence of such a type of expansion in any case is no worse than the convergence of the coupling constant perturbation series. The reason is as follows: being the simplest from the practical point of view the coupling constant perturbation series is the worst in the mathematical sense. The problem is that a small parameter (the coupling constant) is a multiplier at the highest derivative terms of the functional-differential SDEs for the generating functional. This means that the equation is a singularly perturbed one and, consequently, the coupling constant perturbation series is an asymptotic series at best. On the other hand, the proposed expansion is regular in the mathematical sense and, consequently, possesses the better convergence properties. (See also [10] for more details.) In addition, note that, for a model 'zero-dimensional' theory at $n = 0$, for which the functional integral becomes the ordinary one, and the SDEs become ordinary differential equations, such an expansion possesses very good convergence properties [10].

The generating relation (12) of Ward identities in the framework of a given iteration scheme takes the form

$$\frac{\mathrm{i}}{d_l}\partial^2\partial_\nu \frac{\delta G^{(i)}}{\delta J_\nu(x)} = \partial_\nu J_\nu(x) G^{(i-1)} + \int \mathrm{d}x' \,\mathrm{tr}\left\{\eta(x,x')\frac{\delta G^{(i-1)}}{\delta \eta(x,x')} - \frac{\delta G^{(i-1)}}{\delta \eta(x',x)}\eta(x',x)\right\}. \quad (32)$$

Its corollaries, such as equations (14), (16) and (20), should be modified correspondingly. These Ward identities are very useful tools in controlling the gauge invariance. As we can see below, the gauge invariance requirements, which are expressed in the form of Ward identities, impose some strict limitations in a given iteration scheme (see, e.g., discussions after equation (96) and in section 5). The first step calculations, which are considered in this paper, demonstrate their compatibility with the gauge invariance (though some question arises for the solutions with chiral symmetry breaking, see section 6 and the conclusion). At the moment we can say nothing about this problem for higher iteration steps.

Consider in more detail the leading approximation described by equations (23)–(25). It is evident that, for a conservation of Poincaré invariance of the theory, one should assume $V_\mu \equiv 0$. Then from equation (24) it follows that

$$\mathrm{tr}\{\gamma_\mu S^{(0)}(x,x)\} = 0. \quad (33)$$

By definition (see (18)) $S^{(0)}$ is the electron propagator at the leading approximation. Therefore it follows from the above-mentioned Poincaré invariance that $S^{(0)}(x, y) = S^{(0)}(x - y)$, and

$$S^{(0)}(x,x) = S^{(0)}(0) = \int \frac{\mathrm{d}p}{(2\pi)^n} S^{(0)}(p).$$

As was stated above, a regularization is always supposed for all such expressions.

As these conditions have been fulfilled the solution of equation (25) is the free propagator

$$S^{(0)} = S^c = (m - \mathrm{i}\hat{\partial})^{-1}.$$

For this solution condition (33) is equivalent to the requirement imposed on a regularization

$$\int \mathrm{d}p \frac{p_\mu}{m^2 - p^2} = 0.$$

It is very difficult to imagine an invariant regularization for which this condition would not be fulfilled.

So, the generating functional of the leading approximation is

$$G^{(0)} = \exp\{\mathrm{Tr}\, S \star \eta\}. \tag{34}$$

As follows from equation (34), the unique connected Green function of the leading approximation is the electron propagator. Other connected Green functions appear during the following iteration steps. It also follows from equation (34) that the generating functional at the leading approximation does not possess the complete Fermi symmetry. Rather, as follows from the definition of the generating functional, the Fermi symmetry implies on the *full* generating functional the condition

$$\frac{\delta^2 G}{\delta\eta^{\beta\alpha}(y,x)\delta\eta^{\beta'\alpha'}(y',x')} = -\frac{\delta^2 G}{\delta\eta^{\beta'\alpha}(y',x)\delta\eta^{\beta\alpha'}(y,x')}. \tag{35}$$

Evidently condition (35) is not fulfilled for $G^{(0)}$ defined by equation (34). The violation of this condition leads particularly to the violation of the connected structure of the leading approximation two-particle (four-point) electron function, which is

$$S_2^{(0)}(x,y;x',y') \equiv \left.\frac{\delta^2 G^{(0)}}{\delta\eta^{\beta\alpha}(y,x)\delta\eta^{\beta'\alpha'}(y',x')}\right|_{J=\eta=0} = S^c(x-y)S^c(x'-y') \tag{36}$$

a term $S^c(x-y')S^c(x'-y)$ having been omitted.

Such a situation is rather typical for nonperturbative calculational schemes with a bilocal source (for example, for $1/N$ expansion in the bilocal source formalism), but discrepancies of such type are not an obstacle for using these iteration schemes. Rather, condition (35) should be satisfied by the *full* generating functional G which is an *exact* solution of SDEs (6) and (7). It is clear that an approximate solution may not possess all the properties of an exact one. For a given case we have just the same situation. Properties of connectivity and Fermi symmetry of higher Green functions, which are not fulfilled at the first steps of the iteration scheme, are restored at subsequent steps. For example, the structure of the disconnected part of the two-electron function is restored as early as the first step of the iteration scheme (see below). At subsequent steps the correct connected structure of many-electron functions and other corollaries of Fermi symmetry are reconstructed. Such a stepwise reconstruction of exact solution properties is very natural for the given iteration scheme as it is based on an approximation of the generating functional $G(J,\eta)$ in the vicinity of zero. The Green functions are coefficients of the generating functional expansion in the vicinity of zero. Therefore only the lowest functions are well described during the first steps of the approximation—at the leading approximation only the electron propagator. Higher many-particle functions come into play later, at following steps, and relation (35) is fulfilled more and more exactly when we approach the exact solution.

A solution of the first step equation is searched for in the form

$$G^{(1)} = P^{(1)}G^{(0)}$$

where the polynomial $P^{(1)}$ is a solution of equations (30) and (31) at $i = 1$ and $V_\mu \equiv 0$. This solution is

$$P^{(1)} = \frac{1}{2}\eta \star S_2^{(1)} \star \eta + S^{(1)} \star \eta + \frac{1}{2\mathrm{i}} J_\mu \star D_{\mu\nu}^{(1)} \star J_\nu + J_\mu \star F_\mu^{(1)} \star \eta + \mathrm{i}\mathcal{A}_\mu^{(1)} \star J_\mu. \tag{37}$$

In correspondence with the above definitions, $S_2^{(1)}(x,y;x',y')$ is a two-electron function, $S^{(1)}(x-y)$ is a correction to the electron propagator, $D_{\mu\nu}^{(1)}(x-y)$ is a photon propagator,

$F_\mu^{(1)}(x; x', y')$ is a three-point function and $\mathcal{A}_\mu^{(1)}$ is a photon-field vacuum expectation value. The superscript denotes the first-step iteration scheme quantities.

Equations for first-step functions follow from equations (30) and (31) at $i = 1$, $V_\mu \equiv 0$ and they have the form

$$\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{d_l}\partial_\mu\partial_\nu\right) D_{\nu\lambda}^{(1)}(x-y) - \mathrm{i}e\,\mathrm{tr}\{\gamma_\mu F_\lambda^{(1)}(y \mid x, x)\} = g_{\mu\lambda}\delta(x-y) \tag{38}$$

$$\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{d_l}\partial_\mu\partial_\nu\right) F_\nu^{(1)}(x \mid x', y') = e\,\mathrm{tr}\{\gamma_\mu S_2^{(1)}(x, x; x', y')\} \tag{39}$$

$$\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{d_l}\partial_\mu\partial_\nu\right) \mathcal{A}_\nu^{(1)}(x) + \mathrm{i}e\,\mathrm{tr}\{\gamma_\mu S^{(1)}(x, x)\} = 0 \tag{40}$$

$$(\mathrm{i}\hat{\partial} - m) S_2^{(1)}(x, y; x', y') - \mathrm{i}e\gamma_\mu S^c(x-y) F_\mu^{(1)}(x \mid x', y') = \delta(x-y') S^c(x'-y) \tag{41}$$

$$(\mathrm{i}\hat{\partial} - m) F_\lambda^{(1)}(z \mid x, y) = e\gamma_\mu S^c(x-y) D_{\mu\lambda}^{(1)}(x-z) \tag{42}$$

$$(\mathrm{i}\hat{\partial} - m) S^{(1)}(x-y) - \mathrm{i}e\gamma_\mu F_\mu^{(1)}(x \mid x, y) + e\gamma_\mu S^c(x-y)\mathcal{A}_\mu^{(1)}(x) = 0. \tag{43}$$

At first sight the system of equations (38)–(43) seems to be overfilled: six equations for five functions. In fact, one can prove that equation (39) is a corollary of the other equations. Further, if one imposes on $S^{(1)}$ the same condition as for $S^{(0)}$ (see (33)), then the existence of a trivial solution $\mathcal{A}_\mu^{(1)} \equiv 0$ for $\mathcal{A}_\mu^{(1)}$ follows from equation (40). We restrict ourselves to this solution for $\mathcal{A}_\mu^{(1)}$, ignoring all others as violating the Poincaré invariance of the theory.

From equation (42) we obtain

$$F_\lambda^{(1)}(z \mid x, y) = -e \int \mathrm{d}x'\, S^c(x-x')\gamma_\mu S^c(x'-y) D_{\mu\lambda}^{(1)}(x'-z). \tag{44}$$

Substituting (44) into (38), we obtain after simple transformations

$$(D_{\mu\nu}^{(1)})^{-1} = (D_{\mu\nu}^c)^{-1} + \Pi_{\mu\nu} \tag{45}$$

where

$$\Pi_{\mu\nu}(x) = \mathrm{i}e^2\,\mathrm{tr}\{\gamma_\mu S^c(x)\gamma_\nu S^c(-x)\} \tag{46}$$

is a free-electron loop. It follows from the Ward identities that $\Pi_{\mu\nu}$ is transversal in the momentum space:

$$\Pi_{\mu\nu}(k) = \Pi(k^2)\pi_{\mu\nu} \tag{47}$$

where

$$\pi_{\mu\nu} \equiv g_{\mu\nu} - \frac{k_\mu k\nu}{k^2}$$

is the transverse projector. On taking into account equation (47) we finally obtain (in momentum space)

$$D_{\mu\nu}^{(1)}(k) = \frac{1}{-k^2 + \Pi(k^2)}\pi_{\mu\nu} - d_l\frac{k_\mu k_\nu}{(k^2)^2}. \tag{48}$$

Then we obtain from equation (41) a two-electron function:

$$S_2^{(1)}(x, y; x', y') = -S^c(x-y')S^c(x'-y) + \mathrm{i}e^2 \int \mathrm{d}x_1\,\mathrm{d}x_2\, S^c(x-x_1)\gamma_\mu$$
$$\times S^c(x_1-y) D_{\mu\nu}^{(1)}(x_1-x_2) S^c(x'-x_2)\gamma_\nu S^c(x_2-y') \tag{49}$$

and from equation (43) a correction to the electron propagator:

$$S^{(1)}(x-y) = \mathrm{i}e^2 \int \mathrm{d}x_1\,\mathrm{d}x_2\, S^c(x-x_1)\gamma_\mu S^c(x_1-x_2) D_{\mu\nu}^{(1)}(x_1-x_2)\gamma_\nu S^c(x_2-y). \tag{50}$$

Thus we have obtained the expressions for all first-step functions. Note that a disconnected part of the function $S_2^{(1)}$ (see (49)) is the 'missing' disconnected part of the leading approximation two-electron function (36). Hence, as was mentioned earlier, the correct structure of the disconnected part of the two-electron function is reconstructed at the first step of the calculations. At subsequent steps a correct crossing-symmetrical structure of the connected part is also reconstructed.

Before renormalization the formulae obtained are formal expressions due to ultraviolet divergences. The question arises: is there an explicit regularization scheme, which is consistent with the considered iterative expansion? Though in the general case we have no positive answer for this question for any finite iteration step, at the same time we have no reason to suppose the usual well known regularization schemes do not work in our case. In particular, the ultraviolet divergence of the free-electron loop (46) can be treated with dimensional regularization. Note that, at $n = 2$ (two-dimensional electrodynamics), this loop converges in dimensional regularization. At $m = 0$ (Schwinger model)

$$\Pi(k^2) = \frac{e^2}{\pi} \tag{51}$$

and the photon propagator (in transverse gauge) is

$$D_{\mu\nu}^{(1)}(k) = \frac{1}{\frac{e^2}{\pi} - k^2}\pi_{\mu\nu} \tag{52}$$

which coincides with the exact result [11].

4. Renormalization

So far we have considered the unrenormalized theory. To give a physical sense to the quantities to be calculated in a theory with ultraviolet divergences it is necessary to carry out the renormalization procedure.

The Lagrangian of QED for the renormalized fields has the form

$$\mathcal{L}_r = -\frac{Z_3}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{2d_l}(\partial_\mu A_\mu)^2 + Z_1\bar{\psi}(\mathrm{i}\hat{\partial} - m_r + e_r\hat{A})\psi - \delta m\bar{\psi}\psi. \tag{53}$$

Here ψ and A_μ are the renormalized fields, Z_1 and Z_3 are renormalization constants of the spinor and gauge fields, respectively, m_r and e_r are the renormalized mass and charge and δm is the counter-term of electron-mass renormalization. We have taken into account the fact that, due to the gauge invariance and Ward identities, the renormalization constants of the spinor field and of the interaction are equal, $Z_1 = Z_2$, and the longitudinal part of the gauge field is not renormalized.

SDEs for a generating functional of renormalized Green functions are

$$\left[Z_3(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) + \frac{1}{d_l}\partial_\mu\partial_\nu\right]\frac{1}{\mathrm{i}}\frac{\delta G}{\delta J_\nu(x)} + \mathrm{i}e_r Z_1\,\mathrm{tr}\left\{\gamma_\mu\frac{\delta G}{\delta\eta(x,x)}\right\} + J_\mu(x)G = 0 \tag{54}$$

$$\delta(x-y)G + Z_1(\mathrm{i}\hat{\partial} - m_r)\frac{\delta G}{\delta\eta(y,x)} - \delta m\frac{\delta G}{\delta\eta(y,x)} + \frac{e_r Z_1}{\mathrm{i}}\gamma_\mu\frac{\delta^2 G}{\delta J_\mu(x)\delta\eta(y,x)} = \int \mathrm{d}x'\,\eta(x,x')\frac{\delta G}{\delta\eta(y,x')}. \tag{55}$$

Generating relation (12) of the Ward identities for the renormalized generating functional preserves its form under substitution $e \to e_r$. Accordingly, its corollaries (13), (16) and (20) are preserved with the following distinction: all quantities are renormalized ones.

Being applied to our iteration scheme, the renormalization procedure means: the generating functional G, as well as all renormalization constants and counter-terms, have corresponding iteration expansions:

$$Z_j = Z_j^{(0)} + Z_j^{(1)} + \cdots \qquad \delta m = \delta m^{(0)} + \delta m^{(1)} + \cdots .$$

Thus renormalized equations of the leading approximation are

$$\left[Z_3^{(0)}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) + \frac{1}{d_l}\partial_\mu\partial_\nu \right] \frac{1}{\mathrm{i}} \frac{\delta G^{(0)}}{\delta J_\nu(x)} + \mathrm{i}e_r Z_1^{(0)} \operatorname{tr}\left\{ \gamma_\mu \frac{\delta G^{(0)}}{\delta \eta(x,x)} \right\} = 0 \tag{56}$$

$$\delta(x-y)G^{(0)} + Z_1^{(0)}(\mathrm{i}\hat{\partial} - m_r)\frac{\delta G^{(0)}}{\delta \eta(y,x)} - \delta m^{(0)} \frac{\delta G^{(0)}}{\delta \eta(y,x)} + \frac{e_r Z_1^{(0)}}{\mathrm{i}} \gamma_\mu \frac{\delta^2 G^{(0)}}{\delta J_\mu(x)\delta\eta(y,x)} = 0. \tag{57}$$

A solution of these equations is the same leading approximation functional (34) but now S^c is the renormalized free propagator:

$$S^c = (m_r - \mathrm{i}\hat{\partial})^{-1}.$$

At that

$$Z_1^{(0)} = 1 \qquad \delta m^{(0)} = 0$$

and the same condition (33) is imposed for a regularization. Note that the value of constant $Z_3^{(0)}$ remains undefined in the framework of the leading approximation. This constant is fixed at the first step of the iteration scheme when the photon propagator comes into play.

The first-step functional is $G^{(1)} = P^{(1)}G^{(0)}$, and taking into account the above definitions and results of the leading approximations for $P^{(1)}$ we obtain the equations

$$\left[Z_3^{(0)}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) + \frac{1}{d_l}\partial_\mu\partial_\nu \right] \frac{1}{\mathrm{i}} \frac{\delta P^{(1)}}{\delta J_\nu(x)} + \mathrm{i}e_r \operatorname{tr}\left\{ \gamma_\mu \frac{\delta P^{(1)}}{\delta \eta(x,x)} \right\} = -J_\mu(x) \tag{58}$$

$$(\mathrm{i}\hat{\partial} - m_r)\frac{\delta P^{(1)}}{\delta \eta(y,x)} + \frac{e_r}{\mathrm{i}}\gamma_\mu \frac{\delta^2 P^{(1)}}{\delta J_\mu(x)\delta\eta(y,x)} + \frac{e_r}{\mathrm{i}}\gamma_\mu S^c(x-y)\frac{\delta P^{(1)}}{\delta J_\mu(x)}$$
$$= \int \mathrm{d}x_1\, \eta(x,x_1)S^c(x_1-y) + Z_1^{(1)}\delta(x-y) + \delta m^{(1)} S^c(x-y). \tag{59}$$

A solution to these equations has the same form as (37) with a distinction: the coefficient functions of polynomial (37), which define the first-step Green functions, are now the renormalized quantities. We do not write out equations for the coefficient functions which coincide with the above unrenormalized equations (38)–(43) up to evident modifications.

For the three-pointer $F_\mu^{(1)}$ we obtain the same expression (44), and for the photon propagator in the momentum space we have

$$D_{\mu\nu}^{(1)}(k) = \frac{1}{-Z_3^{(0)}k^2 + \Pi(k^2)}\pi_{\mu\nu} - d_l \frac{k_\mu k_\nu}{(k^2)^2} \tag{60}$$

where Π is the same free-electron loop (46) and (47) with a substitution $e \to e_r$, $m \to m_r$.

Expression (49) for the two-electron function $S_2^{(1)}$ is not altered (with the same substitution again). For the first-step correction to the electron propagator we have

$$S^{(1)}(x-y) = \mathrm{i}e_r^2 \int \mathrm{d}x_1\, \mathrm{d}x_2\, S^c(x-x_1)\gamma_\mu S^c(x_1-x_2)D_{\mu\nu}^{(1)}(x_1-x_2)\gamma_\nu S^c(x_2-y)$$
$$+ Z_1^{(1)} S^c(x-y) - \delta m^{(1)} \int \mathrm{d}x_1\, S^c(x-x_1)S^c(x_1-y). \tag{61}$$

If the photon propagator (60) possesses a pole in the point $k^2 = \mu^2$, which corresponds to a particle with mass μ, then for $\Pi(k^2)$ the following normalization conditions should be satisfied:

$$\Pi(\mu^2) = Z_3^{(0)}\mu^2 \qquad \Pi'(\mu^2) = Z_3^{(0)} - 1. \tag{62}$$

In the two-dimensional case ($n = 2$)

$$\Pi(k^2) = -\frac{e_r^2 k^2}{\pi}\int_0^1 \mathrm{d}x \frac{x(1-x)}{m_r^2 - x(1-x)k^2 - \mathrm{i}0}. \tag{63}$$

In particular, at $m_r = 0$: $\Pi(k^2) = e_r^2/\pi$, and normalization conditions (62) give $Z_3^{(0)} = 1$, $\mu^2 = e_r^2/\pi$, i.e. the result for the Schwinger model is unchanged by the renormalization.

For four-dimensional theory $\Pi(k^2)$ is an ultraviolet-divergent quantity. At the dimensional regularization ($n = 4 - 2\epsilon$)

$$\Pi(k^2) = -\frac{e_r^2 k^2}{2\pi^2}(2\pi)^\epsilon \Gamma(\epsilon)\int_0^1 \mathrm{d}x \frac{x(1-x)}{(m_r^2 - x(1-x)k^2 - \mathrm{i}0)^\epsilon}. \tag{64}$$

Here $\Gamma(x)$ is a gamma function. In this case the normalization conditions give us the following:

$$Z_3^{(0)} = 1 - \frac{\alpha_r}{3\pi}\left\{\frac{1}{\epsilon} + \psi(1) + \log 2\pi - \log\frac{m_r^2}{M^2}\right\} \qquad \mu^2 = 0.$$

Here $\alpha_r = e_r^2/4\pi$, $\psi(1)$ is the Euler constant and M^2 is a mass parameter of the dimensional regularization.

In the Euclidean region $k^2 < 0$ the renormalized photon propagator in four-dimensional theory possesses a nonphysical pole at $k^2 \simeq -m_r^2 \exp\{\frac{3\pi}{\alpha_r}\}$—it is the Landau pole [12]. This pole also arises in the renormalization group summation [1] and in the framework of the $1/N$ expansion [13]. The presence of this pole is a serious difficulty and, in particular, it prevents one from making sensible calculations (or needs some refined construction which seems to be superfluous for QED [3]). In coupling constant perturbation theory one can exploit a small α_r value to avoid the troubles, but in the nonperturbative scheme used, the unique noncontradictory possibility is the choice $\alpha_r = 0$, and the theory becomes trivial.

Therefore, as a result of the investigation of the first-step equation of our calculational scheme, the following conclusion can be drawn: for the four-dimensional QED the renormalization of theory leads to triviality as early as the first step of the calculations. This result, on the one hand, demonstrates that the proposed scheme correctly obtains the nonperturbative content of QED, but, on the other hand, such a version of the scheme is practically useless in contrast to the usual perturbation theory. However, the other version of the same calculational scheme exists which permits one to make sensible nonperturbative calculations in QED. To state this version is the concern of the remainder of this paper.

5. Nonperturbative vacuum

The iteration scheme considered above can be modified in such a manner that it becomes insensitive to triviality, at least for the first nontrivial steps of the calculation, i.e. applicable to nonperturbative calculations in QED. It means a possibility to calculate physical quantities without contradiction with the triviality of theory, in a similar way it is done in the perturbation theory for the renormalized coupling constant. Of course, one cannot say that the situation is fully analogous with coupling constant perturbation theory. In the framework of perturbation theory one can be sure that calculations in any finite order do not lead to pathologies. It would be untimely to allege the same statement for the given iteration scheme. One cannot exclude,

at the moment, the possibility of a pathology (which originates from the triviality of four-dimensional QED) in a higher step of the iterations. In the case of such a pathology, the use of a grand unification for its elimination seems to be inevitable. A full answer for these major questions can be given only with detailed investigation of higher iteration steps. Nevertheless, as we shall see, an analysis of the first steps of a given iteration scheme demonstrates that it includes in a natural way such well known nonperturbative treatments of QED as the ladder and rainbow approximations.

To pass this modification of the iteration scheme (which we shall call calculations over nonperturbative vacuum) let us resolve SDE (6) with regard to the first derivative of the generating functional with respect to J_μ:

$$\frac{1}{\mathrm{i}}\frac{\delta G}{\delta J_\mu(x)} = -\int \mathrm{d}x_1\, D^c_{\mu\nu}(x-x_1)\left\{J_\nu(x_1)G + \mathrm{i}e\,\mathrm{tr}\,\gamma_\nu\frac{\delta G}{\delta\eta(x_1,x_1)}\right\} \tag{65}$$

and substitute it into the second SDE (7). As a result we obtain the 'integrated over A_μ' (in the functional-integral terminology) equation

$$\begin{aligned}\delta(x-y)G + (\mathrm{i}\hat\partial - m)\frac{\delta G}{\delta\eta(y,x)} + \frac{e^2}{\mathrm{i}}\int \mathrm{d}x_1\, D^c_{\mu\nu}(x-x_1)\gamma_\mu\frac{\delta}{\delta\eta(y,x)}\,\mathrm{tr}\,\gamma_\nu\frac{\delta G}{\delta\eta(x_1,x_1)}\\ = \int \mathrm{d}x_1\left\{\eta(x,x_1)\frac{\delta G}{\delta\eta(y,x_1)} + eD^c_{\mu\nu}(x-x_1)J_\nu(x_1)\gamma_\mu\frac{\delta G}{\delta\eta(y,x)}\right\}.\end{aligned} \tag{66}$$

Exploiting the Fermi-symmetry condition (35) let us transform equation (66) in the following manner:

$$\begin{aligned}\delta(x-y)G + (\mathrm{i}\hat\partial - m)\frac{\delta G}{\delta\eta(y,x)} + \mathrm{i}e^2\int \mathrm{d}x_1\, D^c_{\mu\nu}(x-x_1)\gamma_\mu\frac{\delta}{\delta\eta(x_1,x)}\gamma_\nu\frac{\delta G}{\delta\eta(y,x_1)}\\ = \int \mathrm{d}x_1\left\{\eta(x,x_1)\frac{\delta G}{\delta\eta(y,x_1)} + eD^c_{\mu\nu}(x-x_1)J_\nu(x_1)\gamma_\mu\frac{\delta G}{\delta\eta(y,x)}\right\}.\end{aligned} \tag{67}$$

From the point of view of *exact* solutions equations (66) and (67) are fully equivalent since the transition from equation (66) to (67) is, in essence, an identical transformation. However, it is not the case for the used iteration scheme since, as was stated earlier, condition (35) is fulfilled only approximately at any finite step of the iteration scheme. Therefore, equations (66) and (67) lead to different expansions. Equation (66) leads, in essence, to the same calculational scheme as above, but equation (67) gives us a fruitful scheme for calculation of physical quantities.

Before writing down the equations of the scheme, note that the photon propagator is not contained directly in the coefficient functions of the polynomials $P^{(i)}$ of the iteration scheme for equations (66) and (67), since these equations are, as pointed out above, the result of 'integration' over the photon variable A_μ. For calculation of the photon propagator and related quantities one should exploit the Dyson formula:

$$\begin{aligned}D_{\mu\nu}(x-y) &\equiv \mathrm{i}\left.\frac{\delta^2 G}{\delta J_\nu(y)\delta J_\mu(x)}\right|_{J=\eta=0}\\ &= D^c_{\mu\nu}(x-y) + \mathrm{i}e\int \mathrm{d}x_1\, D^c_{\mu\rho}(x-x_1)\,\mathrm{tr}\,\gamma_\rho F_\nu(y;x_1,x_1)\end{aligned} \tag{68}$$

which is a consequence of a single differentiation of equation (65).

In correspondence with the general principle of iteration construction we choose as the leading approximation equations (66) and (67) with $J_\mu = 0$, $\eta = 0$ in their coefficients. These equations are

$$\delta(x-y)G^{(0)} + (\mathrm{i}\hat\partial - m)\frac{\delta G^{(0)}}{\delta\eta(y,x)} + \frac{e^2}{\mathrm{i}}\int \mathrm{d}x_1\, D^c_{\mu\nu}(x-x_1)\gamma_\mu\frac{\delta}{\delta\eta(y,x)}\,\mathrm{tr}\,\gamma_\nu\frac{\delta G^{(0)}}{\delta\eta(x_1,x_1)} = 0 \tag{69}$$

and

$$\delta(x-y)G^{(0)} + (\mathrm{i}\hat{\partial} - m)\frac{\delta G^{(0)}}{\delta\eta(y,x)} + \mathrm{i}e^2\int \mathrm{d}x_1\, D^c_{\mu\nu}(x-x_1)\gamma_\mu \frac{\delta}{\delta\eta(x_1,x)}\gamma_\nu \frac{\delta G^{(0)}}{\delta\eta(y,x_1)} = 0. \tag{70}$$

The functional

$$G^{(0)} = \exp\{\mathrm{Tr}\, S^{(0)} \star \eta\} \tag{71}$$

is a solution for both these equations but, whereas the solution of the characteristic equation corresponding to equation (69) is the free propagator $S^{(0)} = S^c$ with condition (33), a characteristic equation corresponding to equation (70) has the form

$$[S^{(0)}]^{-1}(x) = (m - \mathrm{i}\hat{\partial})\delta(x) - \mathrm{i}e^2 D^c_{\mu\nu}(x)\gamma_\mu S^{(0)}(x)\gamma_\nu \tag{72}$$

i.e. it is a nontrivial nonlinear equation for $S^{(0)}$, and so we call this scheme the calculations over nonperturbative vacuum, as distinct from the above scheme of equation (66), which is based on the free solution S^c.

A detailed discussion of equation (72) is given below. At the moment we note that, at $m = 0$ (chiral limit) in the transverse gauge $d_l = 0$, this equation possesses a simple solution:

$$S^{(0)} = -1/\mathrm{i}\hat{\partial}. \tag{73}$$

In fact, in coordinate space† at $n > 2$

$$D^c_{\mu\nu}(x) = \frac{\mathrm{e}^{-\mathrm{i}\pi n/2}\Gamma(\frac{n}{2}-1)}{4\mathrm{i}\pi^{n/2}(x^2-\mathrm{i}0)^{n/2-1}}\left[\frac{1+d_l}{2}g_{\mu\nu} + (1-d_l)\left(\frac{n}{2}-1\right)\frac{x_\mu x_\nu}{x^2-\mathrm{i}0}\right]. \tag{74}$$

In the transverse gauge $d_l = 0$ the function $D^c_{\mu\nu}(x)$ possesses an important property ('$\hat{x}$ transversality')

$$D^c_{\mu\nu}(x)\gamma_\mu\hat{x}\gamma_\nu = 0. \tag{75}$$

From this property the existence of solution (73) at $m = 0$ follows immediately.

At $m \neq 0$ the solution of the characteristic equation is a rather complicated problem which needs application of some approximate or numerical methods. The search for solutions, breaking the chiral symmetry, is also of great interest (see below, section 6).

An iteration equation for the nonperturbative vacuum in correspondence with equations (67) and (70) is

$$\delta(x-y)G^{(i)} + (\mathrm{i}\hat{\partial} - m)\frac{\delta G^{(i)}}{\delta\eta(y,x)} + \mathrm{i}e^2\int \mathrm{d}x_1\, D^c_{\mu\nu}(x-x_1)\gamma_\mu \frac{\delta}{\delta\eta(x_1,x)}\gamma_\nu \frac{\delta G^{(i)}}{\delta\eta(y,x_1)}$$
$$= \int \mathrm{d}x_1 \left\{\eta(x,x_1)\frac{\delta G^{(i-1)}}{\delta\eta(y,x_1)} + eD^c_{\mu\nu}(x-x_1)J_\nu(x_1)\gamma_\mu\frac{\delta G^{(i-1)}}{\delta\eta(y,x)}\right\}. \tag{76}$$

A solution of the first-step equation is $G^{(1)} = P^{(1)}G^{(0)}$, where

$$P^{(1)} = \tfrac{1}{2}\eta \star S^{(1)}_2 \star \eta + S^{(1)} \star \eta + J_\mu \star F^{(1)}_\mu \star \eta. \tag{77}$$

Taking into account the leading approximation (70) and the characteristic equation (72) we obtain for the three-point function $F^{(1)}_\lambda$, the two-electron function $S^{(1)}_2$ and the propagator correction $S^{(1)}$ the following equations:

$$F^{(1)}_\lambda(z;x,y) = -e\int \mathrm{d}x_1\, D^c_{\lambda\mu}(z-x_1)S^{(0)}(x-x_1)\gamma_\mu S^{(0)}(x_1-y) + \mathrm{i}e^2$$

† The case $n = 2$ is discussed below, see the conclusion.

$$\times \int \mathrm{d}x_1\,\mathrm{d}y_1\, D^c_{\mu\nu}(x_1-y_1)S^{(0)}(x-x_1)\gamma_\mu F^{(1)}_\lambda(z;x_1,y_1)\gamma_\nu S^{(0)}(y_1-y) \quad (78)$$

$$S^{(1)}_2(x,y;x',y') = -S^{(0)}(x-y')S^{(0)}(x'-y) + \mathrm{i}e^2\int \mathrm{d}x_1\,\mathrm{d}y_1\, D^c_{\mu\nu}(x_1-y_1)$$
$$\times S^{(0)}(x-x_1)\gamma_\mu S^{(1)}_2(x_1,y_1;x',y')\gamma_\nu S^{(0)}(y_1-y) \quad (79)$$

$$S^{(1)}(x-y) = \mathrm{i}e^2\int \mathrm{d}x_1\,\mathrm{d}y_1\, D^c_{\mu\nu}(x_1-y_1)S^{(0)}(x-x_1)\gamma_\mu S^{(1)}_2(x_1,y_1;y_1,y)\gamma_\nu + \mathrm{i}e^2$$
$$\times \int \mathrm{d}x_1\,\mathrm{d}y_1\, D^c_{\mu\nu}(x_1-y_1)S^{(0)}(x-x_1)\gamma_\mu S^{(1)}(x_1-y_1)\gamma_\nu S^{(0)}(y_1-y). \quad (80)$$

The first-step equations (78)–(80) are much more complicated in comparison with the first-step equations over perturbative vacuum considered above. In diagram language, perturbative vacuum equations (38)–(43) correspond to the summation of 'chains', and its solutions can be easily presented in a general form (see (44)–(50)). Equations (78)–(80) and the characteristic equation (72) in diagram language correspond to the well known ladder approximation. Such equations for separate Green functions have been repeatedly presented and investigated in the literature (see [14–16], and also [17] and references therein) as the simplest nonperturbative approximations for the exact Dyson equations [18] which constitute an infinite system of engaging equations. In our treatment, in contrast to earlier investigations, these equations are not a result of the more or less arbitrary truncation of Dyson equations, but are a consistent part of the iteration scheme. This condition is very important for solving such problems as taking into account requirements of gauge invariance and renormalizability, which often became stumbling blocks in the investigation of nonperturbative approximations. So, for example, one can often encounter the statement that equation (72) for the electron propagator contradicts the Ward identity (16) and, consequently, is not consistent with gauge invariance (see a discussion of this problem in [17]). However, one should not forget that a comparison of an *approximate* equation (72) with the *exact* Ward identity (16) is not correct, and when an iteration scheme is absent the formulation of this problem, strictly speaking, is incorrect itself. In the framework of our iteration scheme this problem is solved very simply. As follows from equation (32), the Ward identity $F^{(i)}_\mu$ in this framework has the form

$$\frac{\mathrm{i}}{d_l}\partial^2\partial_\nu F^{(i)}_\nu(x;x',y') = e[\delta(x-y')-\delta(x-x')]S^{(i-1)}(x'-y'). \quad (81)$$

It is easy to prove that, for $i=1$, the relation (81) and equation (72) are consistent with the first-step equation (78) for $F^{(1)}_\mu$, i.e. the requirement of gauge invariance is fulfilled.

Now we turn to the renormalization. A renormalized equation for $\delta G/\delta J_\mu$ has the form

$$\frac{1}{\mathrm{i}}\frac{\delta G}{\delta J_\mu(x)} = -\int \mathrm{d}x_1\, D^c_{\mu\nu}(x-x_1 \mid Z_3)\left\{J_\nu(x_1)G + Z_1\mathrm{i}e_r\,\mathrm{tr}\,\gamma_\nu\frac{\delta G}{\delta\eta(x_1,x_1)}\right\} \quad (82)$$

where

$$D^c_{\mu\nu}(Z_3) = \left[Z_3(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu) + \frac{1}{d_l}\partial_\mu\partial_\nu\right]^{-1}.$$

Correspondingly, the renormalized SDE (67) is

$$\delta(x-y)G + Z_1(\mathrm{i}\hat{\partial} - m_r)\frac{\delta G}{\delta\eta(y,x)} - \delta m\frac{\delta G}{\delta\eta(y,x)}$$
$$+\mathrm{i}(e_rZ_1)^2\int \mathrm{d}x_1\, D^c_{\mu\nu}(x-x_1 \mid Z_3)\gamma_\mu\frac{\delta}{\delta\eta(x_1,x)}\gamma_\nu\frac{\delta G}{\delta\eta(y,x_1)}$$
$$= \int \mathrm{d}x_1\left\{\eta(x,x_1)\frac{\delta G}{\delta\eta(y,x_1)} + Z_1eD^c_{\mu\nu}(x-x_1 \mid Z_3)J_\nu(x_1)\gamma_\mu\frac{\delta G}{\delta\eta(y,x)}\right\}. \quad (83)$$

A renormalized leading approximation equation is

$$\delta(x-y)G^{(0)} + Z_1^{(0)}(\mathrm{i}\hat{\partial} - m_r)\frac{\delta G^{(0)}}{\delta\eta(y,x)} - \delta m^{(0)}\frac{\delta G^{(0)}}{\delta\eta(y,x)} + \mathrm{i}(e_r Z_1^{(0)})^2 \int \mathrm{d}x_1\, D^c_{\mu\nu}(x-x_1 \mid Z_3^{(0)})\gamma_\mu \frac{\delta}{\delta\eta(x_1,x)}\gamma_\nu \frac{\delta G^{(0)}}{\delta\eta(y,x_1)} = 0. \tag{84}$$

Equation (84) has a solution in the form of the same functional (71), where $S^{(0)}$ is a solution of the renormalized characteristic equation:

$$[S^{(0)}]^{-1}(x) = (Z_1^{(0)}(m_r - \mathrm{i}\hat{\partial}) + \delta m^{(0)})\delta(x) - \mathrm{i}(e_r Z_1^{(0)})^2 D^c_{\mu\nu}(x \mid Z_3^{(0)})\gamma_\mu S^{(0)}(x)\gamma_\nu. \tag{85}$$

A renormalized photon propagator is defined by the Dyson formula

$$D_{\mu\nu}(x-y) = D^c_{\mu\nu}(x-y \mid Z_3) + \mathrm{i}e_r Z_1 \int \mathrm{d}x_1\, D^c_{\mu\rho}(x-x_1 \mid Z_3)\,\mathrm{tr}\,\gamma_\rho F_\nu(y;x_1,x_1). \tag{86}$$

Since $F_\nu^{(0)} \equiv 0$ at the leading approximation, then

$$D^{(0)}_{\mu\nu} = D^c_{\mu\nu}(Z_3^{(0)}) = \frac{1}{Z_3^{(0)}\partial^2}\left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right) + d_l\frac{\partial_\mu\partial_\nu}{(\partial^2)^2}.$$

From the normalization condition for the photon propagator we obtain

$$Z_3^{(0)} = 1.$$

Therefore, in all the above formulae one can replace $D^c_{\mu\nu}(Z_3^{(0)})$ with $D^c_{\mu\nu}$.

In the general case the renormalized photon propagator in momentum space is

$$D_{\mu\nu}(k) = D^c_{\mu\nu}(k \mid Z_3) + \mathrm{i}e_r Z_1 D^c_{\mu\rho}(k \mid Z_3)\int \frac{\mathrm{d}p}{(2\pi)^n}\,\mathrm{tr}\{\gamma_\rho F_\nu(k;p)\}$$

where $F_\nu(k;p)$ is a Fourier image of the three-pointer:

$$F_\nu(z;x,y) = \int \frac{\mathrm{d}p}{(2\pi)^n}\frac{\mathrm{d}q}{(2\pi)^n}\frac{\mathrm{d}k}{(2\pi)^n}\mathrm{e}^{-\mathrm{i}px+\mathrm{i}qy-\mathrm{i}kz}(2\pi)^n\delta(p-q+k)F_\nu(k;p). \tag{87}$$

The transversality follows from the Ward identities:

$$\mathrm{i}e_r Z_1 \int \frac{\mathrm{d}p}{(2\pi)^n}\,\mathrm{tr}\{\gamma_\rho F_\nu(k;p)\} = \pi_{\rho\nu} f(k^2)$$

and, taking into account the given definition of function $f(k^2)$, we obtain

$$D_{\mu\nu}(k) = -\frac{1+f(k^2)}{Z_3 k^2}\pi_{\mu\nu} - d_l\frac{k_\mu k_\nu}{(k^2)^2}. \tag{88}$$

The zero-mass normalization condition ('photon normalization') gives us the following:

$$Z_3 = 1 + f(0). \tag{89}$$

The constant Z_1 and the mass-renormalization counter-term δm are connected with the normalization of the electron propagator. In the general case the electron propagator in the momentum space is defined by two scalar functions:

$$S(p) = \frac{1}{b(p^2) - a(p^2)\hat{p}} = \frac{b(p^2) + a(p^2)\hat{p}}{b^2(p^2) - a^2(p^2)p^2}.$$

If the propagator $S(p)$ possesses a pole at the point $p^2 = m_r^2$, then the normalization conditions in this point are

$$b(m_r^2) = m_r a(m_r^2) \tag{90}$$

$$a(m_r^2) + 2m_r^2 a'(m_r^2) - 2m_r b'(m_r^2) = 1. \tag{91}$$

A chiral limit for the renormalized theory means that the chiral nonsymmetric terms disappear from the renormalized Lagrangian (53):

$$Z_1 m_r + \delta m \to 0. \tag{92}$$

For the transverse gauge $d_l = 0$ the renormalized characteristic equation (85) has a chiral symmetric solution at the chiral limit

$$S^{(0)} = -\frac{1}{Z_1^{(0)}\hat{p}}. \tag{93}$$

For this solution normalization conditions (90) and (91) give us the relations

$$Z_1^{(0)} = 1 \qquad m_r = \delta m^{(0)} = 0.$$

Hence, this solution coincides with the unrenormalized solution (73).

Renormalized equations for the first-step coefficient functions are

$$\begin{aligned} F_\lambda^{(1)}(z; x, y) = {} & -e_r Z_1^{(0)} \int \mathrm{d}x_1\, D^c_{\lambda\mu}(z - x_1) S^{(0)}(x - x_1)\gamma_\mu S^{(0)}(x_1 - y) \\ & + \mathrm{i}(e_r Z_1^{(0)})^2 \int \mathrm{d}x_1\, \mathrm{d}y_1\, D^c_{\mu\nu}(x_1 - y_1) S^{(0)}(x - x_1)\gamma_\mu \\ & \times F_\lambda^{(1)}(z; x_1, y_1)\gamma_\nu S^{(0)}(y_1 - y) \end{aligned} \tag{94}$$

$$\begin{aligned} S_2^{(1)}(x, y; x', y') = {} & -S^{(0)}(x - y')S^{(0)}(x' - y) + \mathrm{i}(e_r Z_1^{(0)})^2 \int \mathrm{d}x_1\, \mathrm{d}y_1\, D^c_{\mu\nu}(x_1 - y_1) \\ & \times S^{(0)}(x - x_1)\gamma_\mu S_2^{(1)}(x_1, y_1; x', y')\gamma_\nu S^{(0)}(y_1 - y) \end{aligned} \tag{95}$$

$$\begin{aligned} S^{(1)}(x - y) = {} & \mathrm{i}(e_r Z_1^{(0)})^2 \int \mathrm{d}x_1\, \mathrm{d}y_1\, D^c_{\mu\nu}(x_1 - y_1) S^{(0)}(x - x_1)\gamma_\mu S_2^{(1)}(x_1, y_1; y_1, y)\gamma_\nu \\ & + \int \mathrm{d}x_1\, S^{(0)}(x - x_1)\Big\{[Z_1^{(1)}(\mathrm{i}\hat{\partial} - m_r) + \delta m^{(1)}]S^{(0)}(x_1 - y) \\ & + \mathrm{i}e_r^2 Z_1^{(0)} \int \mathrm{d}y_1\, [2Z_1^{(1)} D^c_{\mu\nu}(x_1 - y_1) + Z_1^{(0)} D^c_{\mu\nu}(x_1 - y_1 \mid Z_3^{(1)})] \\ & \times \gamma_\mu S^{(0)}(x_1 - y_1)\gamma_\nu S^{(0)}(y_1 - y)\Big\} + \mathrm{i}(e_r Z_1^{(0)})^2 \int \mathrm{d}x_1\, \mathrm{d}y_1\, D^c_{\mu\nu}(x_1 - y_1) \\ & \times S^{(0)}(x - x_1)\gamma_\mu S^{(1)}(x_1 - y_1)\gamma_\nu S^{(0)}(y_1 - y). \end{aligned} \tag{96}$$

As we can see, the gauge invariance applies rather strict conditions: the constant $Z_1^{(0)}$ is the unique renormalization constant for two first-step equations (94) and (95). In the equation for $S^{(1)}$ new renormalization constants appear: there is $Z_1^{(1)}$, $Z_3^{(1)}$ (the last one is fixed by condition (89)) and the first-step mass renormalization counter-term $\delta m^{(1)}$.

6. Vertex

Let us move from the three-point function F_λ to the truncated three-pointer—the vertex function (19). In correspondence with equation (94) the first-step equation for the vertex is

$$\begin{aligned} \Gamma_\lambda(z; x, y) = {} & -e\gamma_\lambda \delta(x - y)\delta(x - z) \\ & + \mathrm{i}e^2 D^c_{\mu\nu}(x - y) \int \mathrm{d}x_1\, \mathrm{d}y_1\, \gamma_\mu S(x - x_1)\Gamma_\lambda(z; x_1, y_1)S(y_1 - y)\gamma_\nu. \end{aligned} \tag{97}$$

In this section we shall use the simplified notation:

$$\Gamma_\lambda \equiv \Gamma_\lambda^{(1)} \qquad e \equiv e_r Z_1^{(0)} \qquad S \equiv S^{(0)}.$$

In the momentum space equation (97) has the form

$$\Gamma_\lambda(k;\,p) = -e\gamma_\lambda + \mathrm{i}e^2 \int \frac{\mathrm{d}p'}{(2\pi)^n} D^c_{\mu\nu}(p-p')\gamma_\mu S(p')\Gamma_\lambda(k;\,p')S(p'+k)\gamma_\nu. \tag{98}$$

Here k is the photon momentum and p is the electron momentum (see definition (87) of the Fourier image of the three-pointer). Equation (98) is known in the literature as the Edwards equation [15]. Note again that, in our approach, this equation is not a result of arbitrary truncation, but is a consequence of the iteration scheme equations. the electron propagator S in this equation is a solution of the characteristic equation (85) which defines the leading vacuum approximation.

We consider a solution of the vertex equation (97) at small k with the above-mentioned chiral symmetric propagator (93) in the transverse gauge $d_l = 0$. To solve this equation it is convenient to introduce a function $\Phi_\lambda(k;\,p)$ which is defined by the relation

$$\Gamma_\lambda(k;\,p) \equiv \hat{p}\Phi_\lambda(k;\,p)(\hat{p}+\hat{k}).$$

Expand Φ_λ near $k = 0$:

$$\Phi_\lambda(k;\,p) = \Phi_\lambda(p) + k_\rho\Phi_{\lambda\rho}(p) + \cdots.$$

Here

$$\Phi_\lambda(p) \equiv \Phi_\lambda(0;\,p) \qquad \Phi_{\lambda\rho}(p) \equiv \partial^k_\rho\Phi_\lambda(k;\,p)|_{k=0}.$$

Then

$$\Gamma_{\lambda\rho}(p) \equiv \partial^k_\rho\Gamma_\lambda(k;\,p)|_{k=0} = \hat{p}\Phi_{\lambda\rho}(p)\hat{p} + \hat{p}\Phi_\lambda(p)\gamma_\rho. \tag{99}$$

Equations for $\Phi_\lambda(p)$ and $\Phi_{\lambda\rho}$ are

$$\hat{p}\Phi_\lambda(p)\hat{p} = -e\gamma_\lambda + \mathrm{i}e_r^2 \int \frac{\mathrm{d}p'}{(2\pi)^n} D^c_{\mu\nu}(p-p')\gamma_\mu\Phi_\lambda(p')\gamma_\nu \tag{100}$$

and

$$\hat{p}\Phi_{\lambda\rho}(p)\hat{p} = -\hat{p}\Phi_\lambda(p)\gamma_\rho + \mathrm{i}e_r^2 \int \frac{\mathrm{d}p'}{(2\pi)^n} D^c_{\mu\nu}(p-p')\gamma_\mu\Phi_{\lambda\rho}(p')\gamma_\nu. \tag{101}$$

It is more convenient to solve equation (100) in the coordinate space:

$$-\hat{\partial}\Phi_\lambda(x)\hat{\partial} = -e\gamma_\lambda\delta(x) + \mathrm{i}e_r^2 D^c_{\mu\nu}(x)\gamma_\mu\Phi_\lambda(x)\gamma_\nu. \tag{102}$$

Here $D^c_{\mu\nu}(x)$ is defined by equation (74). Due to the property of '$\hat{x}$ transversality' (see (75)) all iterations of equation (102) turn to zero. Indeed, a zero approximation is

$$\Phi^0_\lambda(p) = -e\frac{1}{\hat{p}}\gamma_\lambda\frac{1}{\hat{p}} = e\partial^p_\lambda\frac{1}{\hat{p}}$$

and in x space

$$\Phi^0_\lambda(x) = e\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi n/2}\Gamma(n/2)}{2\pi^{n/2}}\frac{x_\lambda\hat{x}}{(x^2-\mathrm{i}0)^{n/2}} \sim \hat{x}.$$

Consequently

$$\Phi_\lambda = \Phi^0_\lambda$$

and

$$\Gamma_\lambda(p) = -e\gamma_\lambda. \tag{103}$$

Let us go on to solving equation (101) for $\Phi_{\lambda\rho}$. (We consider now the four-dimensional case $n = 4$.) This equation is also more convenient for solving in the coordinate space, where it has the form

$$-\hat{\partial}\Phi_{\lambda\rho}(x)\hat{\partial} = \frac{e}{2\pi^2}\frac{\gamma_\lambda \hat{x}\gamma_\rho}{(x^2 - \mathrm{i}0)^2} + \mathrm{i}e_r^2 D^c_{\mu\nu}(x)\gamma_\mu \Phi_{\lambda\rho}(x)\gamma_\nu. \tag{104}$$

(We have taken into account the result of solving the equation for $\Phi_\lambda(p)$.) Here

$$D^c_{\mu\nu}(x) = \frac{1}{4\mathrm{i}\pi^2(x^2 - \mathrm{i}0)}\left(\frac{g_{\mu\nu}}{2} + \frac{x_\mu x_\nu}{x^2 - \mathrm{i}0}\right). \tag{105}$$

Expanding $\Phi_{\lambda\rho}$ over spinor structures

$$\Phi_{\lambda\rho} = \Phi_{\lambda\rho\sigma}\gamma_\sigma + \Phi^5_{\lambda\rho\sigma}\gamma_5\gamma_\sigma$$

we obtain two equations:

$$\begin{aligned}(g_{\sigma\tau}\partial^2 - 2\partial_\sigma\partial_\tau)\Phi_{\lambda\rho\sigma}(x) &= \frac{e}{2\pi^2}\frac{g_{\lambda\tau}x_\rho - g_{\lambda\rho}x_\tau + g_{\rho\tau}x_\lambda}{(x^2 - \mathrm{i}0)^2} \\ &\quad -\frac{2\alpha_r}{\pi}\frac{1}{x^2 - \mathrm{i}0}\left(g_{\sigma\tau} - \frac{x_\sigma x_\tau}{x^2 - \mathrm{i}0}\right)\Phi_{\lambda\rho\sigma}(x) \qquad (106)\\ (g_{\sigma\tau}\partial^2 - 2\partial_\sigma\partial_\tau)\Phi^5_{\lambda\rho\sigma}(x) &= \frac{\mathrm{i}e}{2\pi^2}\epsilon_{\lambda\rho\sigma\tau}\frac{x_\sigma}{(x^2 - \mathrm{i}0)^2} \\ &\quad -\frac{2\alpha_r}{\pi}\frac{1}{x^2 - \mathrm{i}0}\left(g_{\sigma\tau} - \frac{x_\sigma x_\tau}{x^2 - \mathrm{i}0}\right)\Phi^5_{\lambda\rho\sigma}(x). \qquad (107)\end{aligned}$$

Here

$$\alpha_r = \frac{e_r^2}{4\pi}.$$

As for the equation for Φ_λ iterations of equation (106) for $\Phi_{\lambda\rho\sigma}$ turn to zero, and its solution is

$$\Phi_{\lambda\rho\sigma}(x) = -\frac{e}{4\pi^2}\frac{x_\lambda x_\rho x_\sigma}{(x^2 - \mathrm{i}0)^2}.$$

Equation (107) for $\Phi^5_{\lambda\rho\sigma}$ possesses a solution

$$\Phi^5_{\lambda\rho\sigma}(x) = \frac{\mathrm{i}e}{4\pi(2\pi - \alpha_r)}\epsilon_{\lambda\rho\sigma\tau}\frac{x_\tau}{x^2 - \mathrm{i}0}.$$

Going to the momentum space we finally obtain for $\Gamma_{\lambda\rho}$ (see (99)):

$$\Gamma_{\lambda\rho}(p) = \mathrm{i}e\frac{\alpha_r}{2\pi - \alpha_r}\epsilon_{\lambda\rho\sigma\tau}\gamma_5\gamma_\sigma\frac{p_\tau}{p^2 + \mathrm{i}0}. \tag{108}$$

The matrix element of the vertex function $\Gamma_\lambda(k; p)$ at small k_μ defines two form factors f_1 and f_2:

$$\bar{u}(q)\Gamma_\lambda u(p) \simeq -e_r f_1 \bar{u}(q)\gamma_\lambda u(p) - \frac{e_r}{2m_r}f_2\bar{u}(q)[\gamma_\lambda, \hat{k}]u(p) \tag{109}$$

which are connected respectively with a physical charge and a magnetic moment (see, e.g., [19]). Here $u(p)$ is a solution of the Dirac equation $(\hat{p} - m_r)u(p) = 0$. Comparing with the calculated vertex we obtain at $k = 0$

$$e_r f_1 = e = e_r Z_1^{(0)}.$$

For the normalization on the charge e_r we put $f_1 = 1$ and, consequently,

$$Z_1^{(0)} = 1.$$

This very important fact means that the normalization of the vertex function on the renormalized charge gives, in our approach, the same value of the renormalization constant Z_1 as the wavefunction renormalization condition (91), i.e. our calculational scheme is consistent with requirements which are imposed by gauge invariance.

As for the second form factor f_2, which defines a correction to the magnetic moment, note that the massless particle, of course, has no magnetic moment. However, the calculated first term of the vertex function expansion in k_μ (see (108)) permits making a nonperturbative estimate for the anomalous magnetic moment in the chiral limit.

Let us define the problem more exactly. In the expansion of f_2 in orders of α_r:

$$f_2 = C_1\frac{\alpha_r}{\pi} + C_2\left(\frac{\alpha_r}{\pi}\right)^2 + \cdots$$

the coefficients C_i, starting with the second one, are mass-dependent. (So they even have different signs for the muon and electron, see, e.g., [20].) Nevertheless, one can, in principle, consider the peculiar 'chiral limit' for these coefficients, i.e. their values at $m_r \to 0$. It is this estimate which we are talking about. To obtain this estimate let us make an identical transformation of the second term in equation (109):

$$\frac{1}{m_r}\bar{u}(q)[\gamma_\lambda, \hat{k}]u(p) = \frac{1}{p^2}\bar{u}(q)[\gamma_\lambda, \hat{k}]\hat{p}u(p). \tag{110}$$

For massive particles it is an identity due to the Dirac equation for $u(p)$. However, for the rhs of equation (110) one can use the result (108) which was obtained for massless particles. Comparing coefficients at $\epsilon_{\lambda\rho\sigma\tau}$, we obtain

$$f_2 = \frac{\alpha_r}{2\pi - \alpha_r}. \tag{111}$$

The first term of the expansion in α_r coincides with the Schwinger correction (see, e.g., [20]). At the formal limit $\alpha_r \to \infty$ the full magnetic moment, in correspondence with equation (111), turns to zero. It is interesting to note that a similar screening of the chromomagnetic moments of quarks was pointed out in some models of nonperturbative quantum chromodynamics (QCD) [21] and also in a relativistic quarkonium model [22].

7. DCSB

In this section we consider a problem of DCSB for QED in the framework of the proposed calculational scheme over the nonperturbative vacuum. Note that our present consideration does not cover all aspects of this complicated problem: in particular, we do not consider a question of the connection of DCSB with the triviality problem. A consideration of this and a number of other questions imposes an investigation of the first-step equations for higher Green functions. We limit ourselves to investigation of the leading approximation equation.

A renormalized equation of the leading approximation for the electron propagator has the form

$$S^{-1}(x) = (m - Z\mathrm{i}\hat{\partial})\delta(x) - \mathrm{i}e^2 D^c_{\mu\nu}(x)\gamma_\mu S(x)\gamma_\nu. \tag{112}$$

Here and everywhere in this section we denote

$$S \equiv S^{(0)} \qquad Z \equiv Z_1^{(0)} \qquad m \equiv Z_1^{(0)}m_r + \delta m^{(0)} \qquad e \equiv e_r Z_1^{(0)}$$

and the free propagator $D^c_{\mu\nu}(x)$ is defined by equation (74).

In the general case (if one does not consider parity breaking solutions) S can be represented as

$$S = \mathrm{i}\hat{\partial}A + B$$

where A and B are scalar functions of unique variables. Correspondingly, for the inverse propagator we have

$$S^{-1} = -\mathrm{i}\hat{\partial}a + b$$

where, in the momentum space, the functions a and b are connected with the functions A and B by the relations

$$A = \frac{a}{b^2 - a^2p^2} \qquad B = \frac{b}{b^2 - a^2p^2}. \tag{113}$$

Taking into account equation (74) and properties of Dirac matrices, the characteristic equation (112) in the coordinate space can be represented as a system of two equations:

$$\mathrm{i}\hat{\partial}a = Z\mathrm{i}\hat{\partial}\delta(x) - 2\alpha d_l \frac{\mathrm{e}^{-\mathrm{i}\pi n/2}\Gamma(\frac{n}{2})}{[\pi(x^2 - \mathrm{i}0)]^{n/2-1}} \cdot \mathrm{i}\hat{\partial}A \tag{114}$$

$$b = m\delta(x) + \alpha(1 - n - d_l)\frac{\mathrm{e}^{-\mathrm{i}\pi n/2}\Gamma(\frac{n}{2} - 1)}{[\pi(x^2 - \mathrm{i}0)]^{n/2-1}} \cdot B. \tag{115}$$

Here

$$\alpha \equiv \frac{e^2}{4\pi}.$$

As is evident from equations (114) and (115), two special gauges exist for the characteristic equation (112).

(1) The transverse Landau gauge $d_l = 0$. In this gauge equation (114) becomes trivial and possesses a solution $a = Z\delta(x)$.

(2) The gauge $d_l = 1 - n$. In this gauge† $b = m\delta(x)$ in the case when the product $B(x)\cdot(x^2 - \mathrm{i}0)^{1-n/2}$ is well defined in a sense of the generalized functions, i.e. divergences are absent. Otherwise, in contrast with the Landau gauge, we have an uncertainty of the type $0 \cdot \infty$. Its evaluation is performed by a regularization, and we can only say that in this gauge $b(p^2) = \text{const}$.

In this section we shall consider the four-dimensional case. Multiplying equations (114) and (115) by x^2, we obtain the equations

$$x^2\mathrm{i}\hat{\partial}a = -\frac{2\alpha}{\pi}d_l\mathrm{i}\hat{\partial}A \tag{116}$$

$$x^2 b = -\frac{\alpha}{\pi}(3 + d_l)B. \tag{117}$$

Such a multiplication can be considered as some regularization of potentially singular products in the rhs of equations (114) and (115). Moreover, going to p space and taking into account equation (113), we obtain for $a(p^2)$ and $b(p^2)$ a system of ordinary differential equations:

$$t\frac{\mathrm{d}^2a}{\mathrm{d}t^2} + 3\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\alpha}{2\pi}d_l\frac{a}{b^2 - a^2t} \tag{118}$$

$$t\frac{\mathrm{d}^2b}{\mathrm{d}t^2} + 2\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{\alpha}{4\pi}(3 + d_l)\frac{b}{b^2 - a^2t}. \tag{119}$$

Here $t = p^2$.

This very complicated system of nonlinear differential equations has the simplest form in the transverse Landau gauge $d_l = 0$. Below we shall exploit this gauge. As was mentioned earlier, in this gauge $a = Z$ and the system (118) and (119) is reduced to the equation for b:

$$t\frac{\mathrm{d}^2b}{\mathrm{d}t^2} + 2\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{3\alpha}{4\pi}\frac{b}{b^2 - Z^2t}. \tag{120}$$

† At $n = 4$ it is the Solovev–Yennie gauge [23].

This equation always has the trivial solution $b \equiv 0$, which corresponds to the above-mentioned chiral symmetric solution (93). The existence of nontrivial solutions is also possible, and their asymptotics are not difficult to find. In fact, at $t \to \infty$ two variants are possible:

(a) $b^2 \sim t$;

(b) $b^2 \ll \mid t \mid$.

It is not difficult to prove that the third possibility $b^2 \gg \mid t \mid$ up to logarithms leads to the variant (a).

Consider the variant (b). Then in the ultraviolet region $t \to \infty$ equation (120) is reduced to the Euler equation, and an asymptotic behaviour of $b(t)$ is

$$b \sim t^{-1/2(1-\sqrt{1-3\alpha_r/\pi})} \qquad \text{at} \quad \alpha_r < \pi/3 \tag{121}$$

$$b \sim t^{-1/2} \log t \qquad \text{at} \quad \alpha_r = \pi/3 \tag{122}$$

and

$$b \sim t^{-1/2} \sin\left\{\frac{\sqrt{3\alpha_r/\pi - 1}}{2} \log t\right\} \qquad \text{at} \quad \alpha_r > \pi/3. \tag{123}$$

Note that here $\alpha_r \equiv \frac{e_r^2}{4\pi}$ is the renormalized fine structure constant.

The existence of a critical value

$$\alpha_r = \alpha_c \equiv \pi/3 \tag{124}$$

should be pointed out, at which a change of ultraviolet behaviour regime takes place. The existence of this critical point was first pointed out in [24]. In [4] such a regime transition was connected with DCSB in QED. (See also [7, 17] where this approach is discussed in detail and an extensive bibliography is given. For the latest developments in this approach see [25].) In all these cited works a scheme with ultraviolet cutoff in Euclidean space was exploited. A main calculational ansatz was a linearization of equation (120), which consists of an approximation†

$$\frac{b}{b^2 - t} \approx \frac{b}{m^2 - t}$$

on the rhs of equation (120). Here $m \equiv b(0)$. After such a linearization the equation for $b(t)$ becomes a hypergeometrical equation. Boundary conditions for this equation are defined from the integral equation which is equation (72) in the momentum space‡. An asymptotic behaviour of the solutions of the linearized equation is given by the same formulae (121)–(123), and this fact is the main argument in favour of the linearized version (at least in the ultraviolet region). These solutions, of course, essentially depend on a cutoff parameter which is included in the boundary conditions. Nevertheless, one was successful in demonstrating the existence of a phase transition at the critical point $\alpha_c = \pi/3$. This transition corresponds to DCSB. In the chiral limit under the critical point (weak coupling) only the chiral symmetric solution exists, but beyond the critical point (strong coupling), when the ultraviolet asymptotics become oscillating, a solution with dynamical mass $b \neq 0$ arises, i.e. the spontaneous breaking of chiral symmetry takes place.

We consider this problem in the pseudo-Euclidean Minkowski space for renormalized equation (112)§. At that point we shall use another approach to the investigation of the propagator equation, which, nevertheless, is similar to the approach of the above-mentioned

† Note that in the cutoff scheme $Z = 1$.

‡ An essential point in the formulation of these boundary conditions is the exploiting of the Euclidean version of the theory.

§ This point seems to be essential since detailed investigations of the unrenormalized equation (72) in Euclidean space at $m = 0$ (see, e.g., [26]) demonstrate the existence of a non-pole complex singularity for its solutions, and, hence, the Euclidean rotation seems to be a problem.

works and is also based on a linearization of the nonlinear equation (112). Since we work with renormalized theory, our results are cutoff-independent. In general, they are consistent with the results of linearized unrenormalized theory in Euclidean space up to the unique exception: in the pre-critical region $\alpha_r < \alpha_c$ DCSB is also possible, but under some condition on the value of α_r. An investigation of this condition requires a study of the equation for the three-point function with the nonperturbative electron propagator and is outside the scope of this paper.

We shall solve equation (112) in transverse gauge $d_l = 0$ at $m \neq 0$ by iterations and a foundation for an iterative solution will be an exact solution at $m = 0$:

$$S_0^{-1} = -Z\hat{p}.$$

At that point the linearization consists of a rather natural procedure from the point of view of the iterative solution: starting from the representation of the inverse propagator

$$S^{-1} = S_0^{-1} + \Sigma$$

we approximate

$$S = [S_0^{-1} + \Sigma]^{-1} \approx S_0 - S_0 \star \Sigma \star S_0 = -\frac{\hat{p}}{Z(p^2 + \mathrm{i}0)} - \frac{\hat{p}\Sigma\hat{p}}{Z^2(p^2 + \mathrm{i}0)^2}.$$

It is clear that such an approximation can be fully valid only in the ultraviolet region $|p^2| \gg \mu^2$, where μ^2 is some scale which can be treated as an infrared cutoff. Since a nonperturbative region of QED is the ultraviolet region, the natural supposition is the following: the ultraviolet behaviour is crucial for basic nonperturbative effects, including DCSB.

Using the property of $\hat{x}$ transversality (see (75)) it is easy to prove $\Sigma = 1 \cdot \sigma$, and for the scalar function σ in the coordinate space we obtain the equation

$$\sigma(x^2) = m\delta(x) + \frac{3\alpha_r}{\pi}\frac{1}{x^2 - \mathrm{i}0} \cdot \phi(x^2) \tag{125}$$

where

$$\phi(p^2) \equiv \frac{\sigma(p^2)}{p^2 + \mathrm{i}0}.$$

The four-dimensional delta function $\delta(x)$ in the Minkowski space is a Lorentz-invariant distribution (see, e.g., [27]), and to solve equation (125) it is convenient to use a representation of the delta function as a limit of a sequence of functions in the x^2 variable. As such a representation we choose the formula

$$\delta^4(x) = \frac{\mathrm{i}}{\pi^2}\lim_{\lambda\to 0}\lambda(x^2 - \mathrm{i}0)^{\lambda-2} \tag{126}$$

which can be easily proved with the Fourier transform of the function $(x^2 - \mathrm{i}0)^{\lambda-2}$ (see, e.g., [28]).

A partial solution (a solution of inhomogeneous equations) will be found as

$$\sigma_0(x^2) = C_0(x^2 - \mathrm{i}0)^{\lambda-2}$$

at small λ. As we shall see below, $1/\lambda$ plays the role of a regularization parameter. Making the corresponding Fourier transform (see [28]), we obtain

$$\phi_0(x^2) = \frac{C_0}{4\lambda(1-\lambda)}(x^2 - \mathrm{i}0)^{\lambda-1} \tag{127}$$

and substituting into equation (125) we finally obtain

$$C_0 = -\frac{4\mathrm{i}m}{3\pi\alpha_r}\lambda^2 + \mathcal{O}(\lambda^3).$$

Consequently, the solution of the inhomogeneous equation is

$$\sigma_0(p^2) = -\frac{4\pi m\lambda}{3\alpha_r}. \tag{128}$$

The full solution of equation (125) is a sum of σ_0 and a general solution of the homogeneous equation. A solution of the homogeneous equation is sought for in the form

$$\bar{\sigma}(x^2) = C(x^2 - \mathrm{i}0)^{\beta-2}.$$

Hence $\bar{\phi}(x^2)$ is defined by the same formula (127) with the substitution $\lambda \to \beta$. For the parameter β we obtain the equation

$$\beta(1-\beta) = \frac{3\alpha_r}{4\pi}. \tag{129}$$

If one chooses a solution with $C = 0$, i.e. limits oneself to consideration of the partial solution (128), then normalization conditions (90) and (91) give†

$$\delta m^{(0)} = \frac{3\alpha_r}{4\pi\lambda} m_r \qquad Z = 1$$

and, correspondingly, in this case

$$S^{-1} = -\hat{p} + m_r.$$

In the chiral limit (see (92)), when $m \to 0$, we obtain $m_r = 0$. Therefore this solution is a chiral symmetric one.

But if one permits solutions with $C \neq 0$, the situation is essentially changed. Note at once that the solutions of the homogeneous equation in p space are singular at $p = 0$, but, due to the essence of the approximation made, all subsequent formulae can be interpreted, as was pointed out above, only at $|p^2| \gg \mu^2$, where μ^2 is an infrared cutoff.

So, in dependence on the value of the parameter α_r, we have for the electron mass function:
(1) At $\alpha_r < \alpha_c$ (weak coupling)

$$b(p^2) = -\frac{4\pi m\lambda}{3\alpha_r} + C(p^2 + \mathrm{i}0)^{-\beta} \tag{130}$$

where $\beta = \frac{1}{2}(1 - \sqrt{1 - \alpha_r/\alpha_c})$‡.
(2) At $\alpha_r = \alpha_c = \pi/3$

$$b(p^2) = -\frac{4\pi m\lambda}{3\alpha_r} + \frac{C \log \frac{p^2+\mathrm{i}0}{M^2}}{(p^2+\mathrm{i}0)^{1/2}}. \tag{131}$$

(3) At $\alpha_r > \alpha_c$ (strong coupling)

$$b(p^2) = -\frac{4\pi m\lambda}{3\alpha_r} + \frac{C \sin\{\frac{\omega}{2} \log \frac{p^2+\mathrm{i}0}{M^2}\}}{(p^2+\mathrm{i}0)^{1/2}}. \tag{132}$$

Here

$$\omega = \sqrt{\frac{\alpha_r}{\alpha_c} - 1}.$$

$a = Z$ for any of the three cases.

Below we consider the chiral limit $m \equiv Zm_r + \delta m^{(0)} = 0$. The solution parameters are fixed by normalization conditions (90) and (91) of the electron propagator. If (as in the

† The formula for the counter-term δm demonstrates a role of λ as a parameter of regularization which removes ultraviolet divergence. Therefore, the representation of the delta function by formula (126) can be considered as a special analytical regularization.

‡ We take a root of equation (129) which corresponds to the ultraviolet asymptotics of the exact solution, see (121).

ultraviolet-cutoff scheme [4, 17, 25]) one takes $Z = 1$, then the normalization conditions become

$$b(m_r^2) = m_r \qquad m_r b'(m_r^2) = 0. \tag{133}$$

In the weak coupling case ($\alpha_r < \alpha_c$) it follows from the normalization conditions that either $C = 0$ or $m_r = 0$, i.e. in this case DCSB solutions are absent. In the critical case $\alpha_r = \alpha_c$ a solution with $m_r \neq 0$ is possible which has the form

$$b(p^2) = \frac{m_r^2}{(p^2 + \mathrm{i}0)^{1/2}} \left(1 + \frac{1}{2} \log \frac{p^2 + \mathrm{i}0}{m_r^2}\right). \tag{134}$$

At $\alpha_r > \alpha_c$ (strong coupling) a DCSB solution also exists:

$$b(p^2) = \frac{m_r^2}{(p^2 + \mathrm{i}0)^{1/2}} \left[\cos\left\{\frac{\omega}{2} \log \frac{p^2 + \mathrm{i}0}{m_r^2}\right\} + \frac{1}{\omega} \sin\left\{\frac{\omega}{2} \log \frac{p^2 + \mathrm{i}0}{m_r^2}\right\}\right]. \tag{135}$$

As we can see, these results correspond to those of the unrenormalized theory with a cutoff in the Euclidean space (see [4, 17, 25]). However, for the renormalized theory we have no prior reason to set $Z = 1$. If one refuses this condition, the results in the weak coupling region can be essentially different. In fact, at $Z \neq 1$ the normalization conditions are

$$b(m_r^2) = m_r Z \qquad 2m_r b'(m_r^2) = Z - 1. \tag{136}$$

At $\alpha_r < \alpha_c$, in contrast with the case $Z = 1$, the existence of a DCSB solution becomes possible. It has the form

$$a = Z = \frac{1}{1 + 2\beta} \qquad b = Z m_r \left(\frac{m_r^2}{p^2 + \mathrm{i}0}\right)^{\beta}. \tag{137}$$

At $\alpha_r = \alpha_c$ a solution is

$$b(p^2) = \frac{m_r^2}{(p^2 + \mathrm{i}0)^{1/2}} \left(Z + \left(Z - \frac{1}{2}\right) \log \frac{p^2 + \mathrm{i}0}{m_r^2}\right). \tag{138}$$

and, finally, at $\alpha_r > \alpha_c$

$$b(p^2) = \frac{m_r^2}{(p^2 + \mathrm{i}0)^{1/2}} \left[Z \cos\left\{\frac{\omega}{2} \log \frac{p^2 + \mathrm{i}0}{m_r^2}\right\} + \frac{1}{\omega} (2Z - 1) \sin\left\{\frac{\omega}{2} \log \frac{p^2 + \mathrm{i}0}{m_r^2}\right\}\right]. \tag{139}$$

($a = Z$ for the two last cases, Z is arbitrary.)

Therefore, taking into account the renormalization freedom, DCSB becomes possible in the weak coupling region which corresponds to the physical sector of the theory. Then, however, an important limitation connected with the gauge invariance arises. The necessity for such a limitation follows also from a consideration of the switch-off-interaction limit. Indeed, it follows from equation (137) that, at $\alpha_r \to 0$, $b \to m_r + \mathcal{O}(\alpha_r)$, i.e. either $m_r = 0$ or DCSB does not disappear in the switch-off-interaction limit. The last possibility looks quite strange. A way to avoid this situation can be found if one considers that the entire picture in the weak coupling region is realized only for some finite values of α_r. A condition for such values is a normalization of the vertex on the physical charge e_r:

$$\Gamma_\mu(k = 0;\, p^2 = m_r^2 \mid \alpha_r, Z) = -e_r \gamma_\mu. \tag{140}$$

Due to the gauge invariance the renormalization constant Z is the same as for the electron propagator equation, and substituting the value $Z = Z(\alpha_r)$ from equation (137) we shall obtain an equation for α_r. Solutions of this equation will define 'permitted' values of the charge. A realization of this programme requires solving an equation for the vertex function with propagator (137).

At $\alpha_r \geqslant \alpha_c$ the situation changes in principle: in this case Z (at the leading approximation) is arbitrary, and the normalization on the physical charge fixes Z, but does not lead to additional limitations on the values of α_r.

8. Conclusion

The above results give us a reason to consider the calculations over the nonperturbative vacuum as a quite adequate scheme for calculation of nonperturbative effects in QED. A formulation of the system of equations for Green functions at any step of this calculational scheme is technically as simple as for the coupling constant perturbation theory† and does not require anything except the primitive differentiation. The equations themselves, of course, are much more complicated but, as we can see from the calculation of the vertex function for the chiral symmetric solution (section 5), they possess much more simple solutions in comparison with analogous equations for Green functions in nonperturbative approximations which are based on a more or less arbitrary truncation of the Dyson equations.

As regards the calculations of section 6, which concern the DCSB problem, one of the main questions concerning the linearized equation for the electron propagator is a problem of the gauge dependence of the results, in particular, of a gauge dependence of the critical constant α_c (see [17] and references therein). In our treatment this equation is not a result of an arbitrary truncation of the Dyson equations but a consistent part of the iteration scheme, and this question becomes unavoidable. In this connection we note that the linearization seems to be substantial only in Landau gauge, and for other gauges the nonlinear effects can scarcely be taken into account with such a simple ansatz. As has been mentioned above, the results of section 6 in the strong coupling region coincide in essence with the results of the so-called rainbow approximation, or 'quenched QED with bare vertex' (see [17, 25]). It is no wonder since the basic equation for the electron propagator is the same, and a difference exists in the regularization and renormalization schemes. Generally, quenched QED is an approximation of QED with the effect of pair creation being neglected. For the electron propagator this approximation permits a more general form of the vertex function. For example, the use of the Curtis–Pennington vertex [29] solves the problem of gauge dependence of the critical coupling α_c, though this does not solve all the problems in the strong coupling region (see [30] and references therein). Our approach is not an approximation of QED but an iteration scheme, so we do not have the possibility for such a generalization of the leading order equation for the electron propagator. As a general remark, note that the strongly coupled four-dimensional QED is of purely theoretical interest as a model gauge theory for the investigation of chiral symmetry breaking phenomenon. The results of various approaches (such as the worldline variational approach, high orders of perturbation theory, quenched approximation, etc) essentially differ in the strong coupling region (see [31] for a review), which indicates our present poor understanding of this problem.

In closing we briefly discuss a problem of DCSB for QCD. In contrast to QED, where nonperturbative effects (and DCSB among them) are defined by the ultraviolet region, for asymptotically free QCD a nonperturbative region is the low-energy infrared region. There a mechanism of DCSB in QCD can be even more simple in comparison with QED, since effectively taking into account gluon self-action in the infrared region inevitably leads to the appearance of new dimensional parameters.

As a simple example of a realization of a similar DCSB mechanism by infrared singularities, consider two-dimensional QED. The analogy among the nonperturbative effects of two-dimensional QED and those of four-dimensional QCD was repeatedly pointed out. In spite of the certainly limited nature of such an analogy, an investigation of the nonperturbative

† Note that the coupling constant perturbation theory can be considered as a partial case of the general iteration scheme based on the principles of section 4. To obtain a perturbation theory series it is enough to consider the same iteration scheme with single Grassman fermion sources over a perturbative vacuum which corresponds to the leading approximation $G^{(0)} = 1$ (see also [9, 10]).

dynamics on this example of a much more technically simple QED can be useful for the understanding of QCD in the nonperturbative region.

At $n = 2$ the free propagator $D^c_{\mu\nu}(x)$ (see (74)) entering equation (112) in the general case is infrared-singular, and it is necessary to introduce some infrared cutoff. However, a gauge exists for which such a cutoff is unnecessary. This is the above-mentioned infrared-finite gauge $d_l = 1 - n = -1$. As follows from equation (74), in this gauge

$$D^c_{\mu\nu}(x) = \frac{\mathrm{i}}{2\pi}\frac{x_\mu x_\nu}{x^2 - \mathrm{i}0}$$

and the equation for the electron propagator is simply

$$S^{-1} = (S^c)^{-1} + 2\alpha S. \tag{141}$$

In the momentum space equation (141) is reduced to a system of algebraic equations, and among its solutions a DCSB solution exists. Surely, in the two-dimensional theory, the spontaneous breaking of continuous chiral symmetry is not realized since such a state is unstable in correspondence with the Mermin–Wagner–Coleman theorem. Nevertheless, the fact itself is remarkable from the point of view of the above-mentioned analogy with four-dimensional QCD.

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References

[1] Bogolyubov N N and Shirkov D V 1980 *Introduction to the Theory of Quantized Fields* (New York: Wiley)
[2] Callaway D J E 1988 *Phys. Rep.* C **167** 241
Gockeler M *et al* 1990 *Phys. Lett.* B **251** 567
[3] Espriu D and Tarrach R 1996 *Phys. Lett.* B **383** 482
[4] Fomin P I, Gusynin V P, Miransky V A and Sitenko Yu A 1983 *Riv. Nuovo Cimento* **6** 1
Fomin P I, Gusynin V P and Miransky V A 1986 *Ukr. Fiz. Zh. (Russ. Ed.)* **31** 171
Miransky V A 1985 *Nuovo Cimento* A **90** 149
[5] Kocić A 1993 *Preprint* hep-lat/9312007
[6] Vasilev A N 1978 *Functional Methods in Quantum Field Theory and Statistics* (Leningrad: Leningrad University) (in Russian)
[7] Roberts C D and Williams A G 1994 *Prog. Part. Nucl. Phys.* **33** 477
[8] Nekrasov M L and Rochev V E 1982 *Theor. Math. Phys.* **48** 689
[9] Rochev V E 1997 *J. Phys. A: Math. Gen.* **30** 3671
[10] Rochev V E 1997 *Quantum Chromodynamics: Collisions, Confinement and Chaos* ed H M Fried and B Muller (Singapore: World Scientific) p 354
Rochev V E and Saponov P A 1998 *Int. J. Mod. Phys.* A **13** 3649
Rochev V E 1999 *Proc. 14th Int. Workshop on High Energy Physics and Quantum Field Theory QFTHEP'99 (Moscow, 1999)* ed B B Levchenko and V I Savrin (Moscow: MSU Press) p 572
Rochev V E 1999 *Preprint* hep-th/9911033
[11] Schwinger J 1962 *Phys. Rev.* **128** 2425
[12] Landau L D and Pomeranchuk I Ya 1955 *Dokl. Akad. Nauk Ser. Fiz.* **102** 489
[13] Kirzhnits D A and Linde A D 1978 *Phys. Lett.* B **73** 323
[14] Nambu Y 1950 *Prog. Theor. Phys.* **5** 614
Bethe H and Salpeter E E 1951 *Phys. Rev.* **82** 309
[15] Edwards S F 1953 *Phys. Rev.* **90** 234
[16] Johnson K, Baker M and Willey R 1964 *Phys. Rev.* **136** B1111
[17] Miransky V A 1993 *Dynamical Symmetry Breaking in Quantum Field Theories* (Singapore: World Scientific)
[18] Dyson F 1949 *Phys. Rev.* **75** 1736

[19] Bogolyubov N N and Shirkov D V 1983 *Quantum Fields* (New York: Benjamin)
[20] Akhiezer A I and Berestetskii V B 1969 *Quantum Electrodynamics* (Moscow: Nauka) (in Russian)
[21] Arbuzov B A, Boos E E, Kurennoy S S and Turashvili K Sh 1985 *Sov. J. Nucl. Phys.* **42** 626
[22] Galkin V O, Mishurov A Yu and Faustov R N 1992 *Sov. J. Nucl. Phys.* **55** 1207
Ebert D, Galkin V O and Faustov R N 1998 *Phys. Rev.* D **57** 5663
[23] Solovev L D 1956 *Dokl. Akad. Nauk Ser. Fiz.* **110** 206
Yennie D R, Frautschi S C and Suura H 1961 *Ann. Phys., NY* **13** 379
[24] Maskawa T and Nakajima H 1974 *Prog. Theor. Phys.* **52** 1326
[25] Gusynin V P, Schreiber A W, Sizer T and Williams A G 1999 *Phys. Rev.* D **60** 065007
Kouno H, Hasegawa A, Nakano M and Tuchitani K 1999 *Preprint* hep-ph/9906519
[26] Dragovic B G, Mavlo D P and Filippov A T 1978 *Fizika* **10** 51
Alekseyev A I, Arbuzov B A and Rodionov A Ya 1980 *Theor. Math. Phys.* **42** 291
[27] Bogolyubov N N, Logunov A A, Oksak A I and Todorov I T 1990 *General Principles of Quantum Field Theory (Mathematical Physics and Applied Mathematics vol 10)* (Dordrecht: Kluwer)
[28] Brychkov Yu A and Prudnikov A P 1977 *Integral Transforms of Generalized Functions* (Moscow: Nauka) (in Russian)
[29] Curtis D C and Pennington M R 1991 *Phys. Rev.* D **44** 536
[30] Kizilersu A, Sizer T and Williams A G 2000 *Preprint* hep-ph/0001147
[31] Schreiber A W, Rosenfelder R and Alexandrou C 2000 *Preprint* hep-th/0007182